Problem set 4 solutions

Problem 1: see "Derivation of vorticity equation" in class notes, lecture 8.

Problem 2: see "Kelvin's circulation theorem" in class notes, lecture 8.

Problem 3.

(a) Barotropic Rossby wave dispersion relation (from notes)

$$\omega = k[u] - \frac{\beta k}{k^2+l^2}$$

zonal phase velocity $c_x = \frac{\omega}{k}$

$$c_x = [u] - \frac{\beta}{k^2+l^2}$$

Solve for meridional wavenumber:

$$\frac{\beta}{k^2+l^2} = [u] - c_x$$

$$\beta = \left([u] - c_x\right)\left(k^2+l^2\right)$$

$$l^2 = \frac{\beta}{[u]-c_x} - k^2$$

$$l = \pm\left(\frac{\beta}{[u]-c_x} - k^2\right)^{\frac{1}{2}}$$

for $k \to 0$, $l > 0$ have

$$l = \left(\frac{\beta}{[u]-c_x}\right)^{\frac{1}{2}}$$

at $60°S$, $[u] \sim 22\ ms^{-1}$ (estimate from graph)

$c_x = 10\ ms^{-1}$ (given in problem)

$$\beta = \frac{\partial f}{\partial y} = \frac{1}{R_e}\frac{\partial f}{\partial \phi} = \frac{2\Omega\cos\phi}{R_e}$$

for $\phi = 60°$

$$\beta = 1.15 \times 10^{-11}\ m^{-1}s^{-1}$$

$$l = \left(\frac{1.15\times10^{-11}}{22-10} \; \frac{m^{-1}s^{-1}}{m\,s^{-1}} \right)^{\frac{1}{2}}$$

$$= 9.8\times10^{-7} \; m^{-1}$$

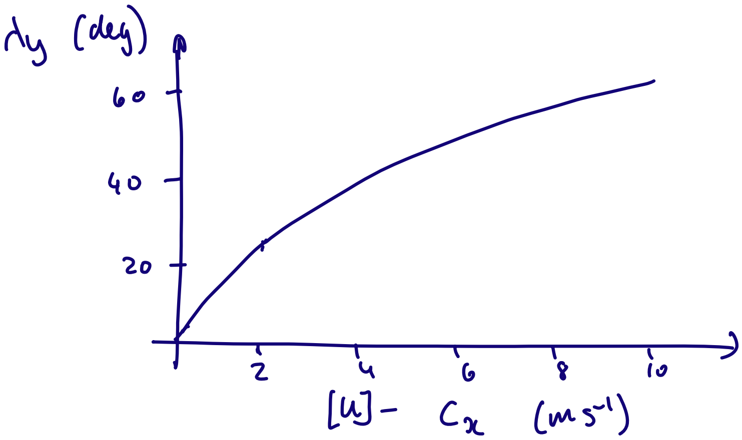
Meridional wavelength: $\lambda_y = \frac{2\pi}{l} = 6.4\times10^{6}\,m$

$\sim 60°$ of latitude.

Rather large!

Note that for $k>0$, l is reduced from this value, implying an even larger wavelength.

Further, note that at $60°$:



Waves that fit in the geometry cannot have $[u]-c$ too large.

$\Rightarrow$ Cannot produce strong wave activity far from $c \simeq [u]$.

(b) WKB theory allows us to assume a wave-like solution which locally satisfies the dispersion relation.

For propagation, we therefore require

$$l^2 > 0$$

This requires $\frac{\beta}{[u]-c_x} - k^2 > 0$

for $k \to 0$

$$\frac{\beta}{[u]-c_x} > 0$$

$$[u]-c_x > 0$$

Critical line occurs when $[u] = c_x$

Waves maintain their phase speeds as they propagate meridionally. For northward propagating wave, trace line vertically from $\phi = 60°S$, $c_x = 10\ m s^{-1}$ until one hits the $[u]$ line.

This occurs at approximately 45°S.

(c) WKB is rather dubious:

$$\lambda_y \gtrsim L$$

where L is the length scale over which $[u]$ varies.

Nevertheless, provides a framework for thinking about eddy propagation.

(d) As wave approaches critical line, $[u]-c_x$ reduces.

From dispersion relation

$$l = \left(\frac{\beta}{[u]-c_x}\right)^{\frac{1}{2}}$$

l becomes large $\rightarrow$ wavelength shortens.

Also remember group velocity

$$C_{gy} = \frac{\partial \omega}{\partial l} = \frac{\partial}{\partial l}\left\{ k[u] - \frac{\beta k}{k^2+l^2} \right\}$$

$$= \frac{2\beta k l}{(k^2+l^2)^2}$$

As critical line is approached, $l \to \infty$

$$C_{gy} \to \frac{2\beta k l}{l^4} = 0$$

Group velocity reduces. Energy can no longer propagate.

→ Wave amplitude grows, eventually becoming nonlinear

→ Wave eventually breaks, dissipates and produces mixing.

Problem 4:

(a) $\Phi^*(x,y,p,t) = A\sin(kx+ly+mp-\omega t)$

$$u_g^* = \frac{+1}{f_0}\frac{\partial\Phi}{\partial y}, \quad v_g^* = \frac{-1}{f_0}\frac{\partial\Phi}{\partial x}$$

$$u_g^* = +\frac{l}{f_0}A\cos(\gamma), \quad v_g^* = -\frac{k}{f_0}A\cos(\gamma)$$

where $\gamma = kx+ly+mp-\omega t$

(b) $$\frac{\partial\Phi}{\partial p} = -\alpha$$

$$\frac{\partial\Phi^*}{\partial p} = -\frac{R_d\,\theta^*\pi}{p}$$

$S = \frac{\sigma R_d\pi}{p}$, $$\frac{1}{S}\frac{\partial\Phi^*}{\partial p} = \frac{p}{\sigma R_d\pi}\frac{\partial\Phi^*}{\partial p} = \frac{p}{\sigma R_d\pi}\frac{R_d\pi}{p}\theta^*$$

$$= \frac{\theta^*}{\sigma}$$

(c) $$\frac{\partial\Phi^*}{\partial p} = mA\cos(\gamma)$$

$$\Rightarrow \quad \theta^* = \frac{\sigma}{S}\frac{\partial\Phi^*}{\partial p} = -\frac{\sigma}{S}mA\cos(\gamma)$$

(d) $$\underset{\sim}{F} = \left(-[u_g^* v_g^*],\ \frac{f_0}{\sigma}[v_g^*\theta^*]\right)$$

$$u_g^* v_g^* = \frac{-A^2kl}{f_0^2}\cos^2(\gamma), \quad -[u_g^* v_g^*] = \frac{+A^2kl}{2f_0^2}$$

$$\frac{f_0}{\sigma}(v_g^*\theta^*) = \frac{f_0}{\sigma}\left(\frac{-k}{f_0}A\right)\left(\frac{-\sigma}{S}mA\right)\cos^2(\gamma), \quad \frac{f_0}{\sigma}[v_g^*\theta^*] = +\frac{kmA^2}{2S}$$

$$\underset{\sim}{F} = \frac{A^2}{2}\left(+\frac{kl}{f_0^2} \;,\; +\frac{km}{S} \right)$$

(e) $\omega = [u]k - \frac{\beta k}{K^2}$, $K^2 = k^2 + l^2 + \frac{m^2 f_0^2}{S}$

$$C_{gy} = \frac{\partial \omega}{\partial l} = \frac{\partial \omega}{\partial K^2}\frac{\partial K^2}{\partial l} = 2l\left(\frac{+\beta k}{K^4} \right)$$

$$= \frac{2\beta}{K^4}(kl)$$

$$C_{gp} = \frac{\partial \omega}{\partial m} = \frac{\partial \omega}{\partial K^2}\frac{\partial K^2}{\partial m} = \frac{f_0^2}{S} 2m \left(\frac{\beta k}{K^4} \right)$$

$$= \frac{2\beta}{K^4}(km)\frac{f_0^2}{S}$$

$$\underset{\sim}{C_g} = \frac{2\beta f_0^2}{K^4}\left(\frac{kl}{f_0^2} \;,\; \frac{km}{S} \right)$$

$$\underset{\sim}{C_g} = \frac{2\beta f_0^2}{K^4}\left(\frac{A^2}{2} \right)^{-1} \underset{\sim}{F}$$

$$\underset{\sim}{C_g} \parallel \underset{\sim}{F}$$

(f) Eliassen-Palm flux is parallel to the Rossby wave group velocity!

EP-flux shows the propagation of waves.