Problem Set 2 solutions

This solution is in height coordinates

Problem 1:

(a) $$\frac{D\gamma}{Dt} = 0$$

$$\frac{\partial \gamma}{\partial t} + \underline{u} \cdot \nabla \gamma = 0 \qquad \cdots (1)$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0 \qquad \cdots (2)$$

Take $\rho(1) + \gamma(2)$:

$$\rho \frac{\partial \gamma}{\partial t} + \rho \underline{u} \cdot \nabla \gamma + \gamma \frac{\partial \rho}{\partial t} + \gamma \nabla \cdot (\rho \underline{u}) = 0$$

$$\underbrace{\rho \frac{\partial \gamma}{\partial t} + \gamma \frac{\partial \rho}{\partial t}}_{\text{product rule}} + \underbrace{\rho \underline{u} \cdot \nabla \gamma + \gamma \nabla \cdot (\rho \underline{u})}_{\text{product rule}} = 0$$

$$\frac{\partial}{\partial t}(\rho \gamma) + \nabla \cdot (\rho \underline{u} \gamma) = 0$$

(b) Take time mean

This should be zonal mean - sorry

$$\frac{\partial}{\partial t}(\overline{\rho \gamma}) + \nabla \cdot (\overline{\rho \underline{u} \gamma}) = 0$$

Neglect ρ' : $\rho \simeq \bar{\rho}$

$$\frac{\partial}{\partial t}(\bar{\rho}\, \bar{\gamma}) + \nabla \cdot (\bar{\rho}\, \overline{u \gamma}) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(\bar{\rho}\, \bar{\gamma}) + \nabla \cdot (\bar{\rho}\, \bar{u}\, \bar{\gamma}) + \nabla \cdot (\bar{\rho}\, \overline{u' \gamma'}) = 0$$

$$\Rightarrow \bar{\rho} \frac{\partial \bar{\gamma}}{\partial t} + \bar{\rho}\, \bar{\underline{u}} \cdot \nabla \bar{\gamma} + \bar{\gamma}\, \nabla \cdot (\bar{\rho}\, \bar{\underline{u}}) = -\nabla \cdot (\bar{\rho}\, \overline{u' \gamma'})$$

Also, time averaged continuity equation:

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot (\overline{\rho \underline{u}}) = 0$$

Neglect ρ'

$$\nabla \cdot (\bar{\rho}\, \bar{\underline{u}}) = 0$$

Hence we may write,

$$\bar{\rho} \frac{\partial \bar{\gamma}}{\partial t} + \bar{\rho}\, \bar{\underline{u}} \cdot \nabla \bar{\gamma} = -\nabla \cdot (\bar{\rho}\, \overline{u'\gamma'})$$

$$\frac{\partial \bar{\gamma}}{\partial t} + \bar{\underline{u}} \cdot \nabla \bar{\gamma} = -\frac{1}{\bar{\rho}} \nabla \cdot (\bar{\rho}\, \overline{u'\gamma'})$$

$$\frac{D \bar{\gamma}}{D t} = -\frac{1}{\bar{\rho}} \nabla \cdot (\bar{\rho}\, \overline{u'\gamma'})$$

as required.

(c) Extra term is the divergence of the eddy flux of the tracer γ.

This term gives the transport of γ effected by eddies rather than the time-mean flow.

So, for example, even if there was no time-mean flow:

$\bar{\underline{u}} = 0$, we could still have that $\frac{\partial \gamma}{\partial t} \neq 0$.

One example of this kind of eddy flux is turbulent diffusion.

→ No net motion of the fluid, but tracer still spreads out.

Problem 2:

(a) Remember $\overline{AB} = \bar{A}\bar{B} + \overline{A'B'}$ .. (1)

and $[AB] = [A][B] + [A^*B^*]$ .. (2)

Take time average of (2)

$$[\overline{AB}] = \overline{[A][B]} + \overline{[A^* B^*]}$$

Now use (1) on first term with $A \to [A]$ and $B \to [B]$

$$[\overline{AB}] = \overbrace{\overline{[A]}\,\overline{[B]}}^{(i)} + \overbrace{\overline{[A]'[B]'}}^{(ii)} + \overbrace{\overline{[A^*B^*]}}^{(iii)} \quad ..(3)$$

(b) In class we divided covariances into:

$$[\overline{AB}] = \underbrace{[\bar{A}][\bar{B}]}_{\text{mean}} + \underbrace{[\bar{A}^*\bar{B}^*]}_{\text{stationary}} + \underbrace{[\overline{A'B'}]}_{\text{transient}} \quad ...(4)$$

In (3) we have:

(i) mean term (as in class and (4))

(ii) zonally symmetric transients (e.g. time variations in the Hadley cell)

(iii) zonally asymmetric eddies (both stationary & transient)

Ultimately both decompositions are valid. The decomposition based on (4) is more popular, partially because it may be related to wave dynamics.

→ Waves with zero phase speed correspond to stationary eddies. Allows for physical interpretation of stationary eddies.

→ Stationary waves/eddies forced by orography / land-ocean contrasts. Unclear what causes zonally symmetric transients.

Problem 4: Problem 3

(a) Geostrophic balance:

$$2\Omega \sin\phi \, v = \frac{1}{a\cos\phi} \frac{\partial \Phi}{\partial \lambda}$$

Take zonal average:

$$2\Omega \sin\phi \, [v] = \frac{1}{a\cos\phi} \left[\frac{\partial \Phi}{\partial \lambda} \right]$$

provided $p < p_s$ at all longitudes, we have

$$2\Omega \sin\phi \, [v] = 0$$

$$[v] = 0$$

Thus, $[v][T] = 0$.

Suppose

$$v = v_0 \cos(\lambda) \quad \cdot\cdot(1)$$
$$T = T_0 + \Delta T \cos(\lambda + \eta) \quad \cdot\cdot(2)$$

(b) $[v] = \frac{1}{2\pi}\int_0^{2\pi} v_0 \cos(\lambda)\, d\lambda = 0$

$$[v][T] = 0$$

(c) $v^* = v$ (since $[v] = 0$)

$T^* = \Delta T \cos(\lambda + \eta)$

$$v^* T^* = v_0 \Delta T \cos(\lambda) \cos(\lambda + \eta)$$

$$[v^* T^*] = \frac{v_0 \Delta T}{2\pi} \int_0^{2\pi} \cos(\lambda) \cos(\lambda + \eta)\, d\lambda$$

Cosine of sum of angles:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$[v^*T^*] = \frac{v_0\Delta T}{2\pi}\int_0^{2\pi}\cos(\lambda)\left\{\cos(\lambda)\cos(\eta) - \sin(\lambda)\sin(\eta)\right\}d\lambda$$

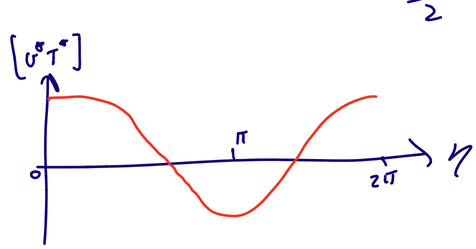
$$= \frac{v_0\Delta T}{2\pi}\left\{\cos(\eta)\overbrace{\int_0^{2\pi}\cos^2(\lambda)\,d\lambda}^{=\pi} - \sin\eta\overbrace{\int_0^{2\pi}\cancel{\cos(\lambda)\sin(\lambda)\,d\lambda}}^{0}\right\}$$

$$[v^*T^*] = \frac{v_0\Delta T}{2}\cos(\eta)$$

For $\eta = 0$ $\quad [v^*T^*] = \frac{v_0\Delta T}{2}$ $\quad$ (max)

For $\eta = \frac{\pi}{2}$ $\quad [v^*T^*] = 0$

For $\eta = \pi$ $\quad [v^*T^*] = -\frac{v_0\Delta T}{2}$ $\quad$ (min)



(d) Geostrophic balance: $\quad 2\Omega\sin\phi\, v = \frac{1}{a\cos\phi}\frac{\partial\Phi}{\partial\lambda}$

Hydrostatic balance: $\quad \frac{\partial\Phi}{\partial p} = -\alpha = -\frac{R_d T}{p}$

$$\Phi(p_s) - \Phi(p) = \int_p^{p_s} -\frac{R_d T}{p}\,dp$$

T is uniform in height:

$$\Phi(p_s) - \Phi(p) = -R_d T\ln\left(\frac{p_s}{p}\right)$$

Now, assume that $\Phi = 0$ at $p = p_s$

$$\Phi(p) = R_d T \ln\left(\frac{p_s}{p}\right)$$

Hence: $$2\Omega \sin\phi\, v = \frac{R_d}{R_e \cos\phi} \ln\left(\frac{p_s}{p}\right) \frac{\partial T}{\partial \lambda}$$

$$= \frac{R_d}{R_e \cos\phi} \ln\left(\frac{p_s}{p}\right) \Delta T \frac{\partial}{\partial \lambda} \cos(\lambda + \eta)$$

$$= \frac{-R_d \Delta T}{R_e \cos\phi} \ln\left(\frac{p_s}{p}\right) \sin(\lambda + \eta)$$

$$v = \frac{-R_d \Delta T}{2\Omega R_e \sin\phi \cos\phi} \ln\left(\frac{p_s}{p}\right) \sin(\lambda + \eta)$$

Now $-\sin(a) = \cos\left(a + \frac{\pi}{2}\right)$

$$v = \left(\frac{R_d \Delta T}{2\Omega R_e \sin\phi \cos\phi}\right) \ln\left(\frac{p_s}{p}\right) \cos\left(\lambda + \frac{\pi}{2} + \eta\right)$$

Compare with (1) above

$$v_0 = \frac{R_d \Delta T}{2\Omega R_e \sin\phi \cos\phi} \ln\left(\frac{p_s}{p}\right)$$

$$\eta = -\frac{\pi}{2}$$

Hence $[v^* T^*] = 0$. (irrespective of latitude or pressure)