Problem set 1 solutions

Problem 1:

At upstream end $u(x \to -\infty) = \mathcal{U}$

By mass conservation, the mass flux through pipe must be constant.

Mass flux $= \rho u(x) w(x) D$

where D is depth of pipe.

At upstream end, have $w(x) = \frac{3W_0}{2}$

$$\Rightarrow \quad \rho u(x) w(x) D = \rho \mathcal{U} \frac{3W_0}{2} D$$

$$u(x) = \frac{3\mathcal{U}}{2}\left(\frac{1}{1 - \frac{1}{2}\tanh(x)}\right)$$

(a) for $x \to \infty$

$$u(x \to \infty) = \frac{3\mathcal{U}}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = 3\mathcal{U}.$$

(b) Rate of change of velocity at $x=0$

$$\frac{Du}{Dt} = \cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x}$$

0

(flow is steady)

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left\{\frac{3\mathcal{U}}{2} \quad \frac{1}{1 - \frac{1}{2}\tanh(x)}\right\}$$

$$= \frac{3\mathcal{U}}{2} \frac{\partial}{\partial v}\left(\frac{1}{1 - v/2}\right) \frac{\partial}{\partial x}(\tanh(x))$$

$$= \frac{3\mathcal{U}}{2}\left(\frac{-1}{2}\right)\left(\frac{-1}{(1 - v/2)^2}\right)\left(1 - \tanh^2(x)\right) \qquad \text{[Google]}$$

$$\frac{\partial u}{\partial x} = \frac{3U}{4} \frac{1}{\left(1-\frac{1}{2}\tanh(x)\right)^2}\left(1-\tanh^2(x)\right)$$

at $x=0$ $\tanh(x)=0$

$$\frac{\partial u}{\partial x} = \frac{3U}{4}$$

$$u(x=0) = \frac{3U}{2}$$

$$\frac{Du}{Dt} = u\frac{\partial u}{\partial x} = \frac{9U^2}{8} = \frac{9}{8}\ ms^{-2}$$

(c) Eulerian vs. Lagrangian viewpoint.

Eulerian view => no change with time $\frac{\partial u}{\partial t} = 0$

Lagrangian view => air parcels accelerate as the pass through the pipe $\frac{Du}{Dt} > 0$

(d) For frictionless flow

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

at $x=0$ $\frac{Du}{Dt} = \frac{9U^2}{8}$

$$\frac{9U^2}{8} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial p}{\partial x} = -\frac{9U^2\rho}{8} = -\frac{9}{8}(1000)\ \ \text{Pa m}^{-1}$$

Problem 2.

$$\frac{D\underline{u}}{Dt} = \frac{D}{Dt}\left\{ u\hat{\underline{\lambda}} + v\hat{\underline{\phi}} + w\hat{\underline{r}} \right\}$$

$$= \frac{Du}{Dt}\hat{\underline{\lambda}} + u\frac{D\hat{\underline{\lambda}}}{Dt} + \dots \qquad \text{(product rule)}$$

Want $$\frac{D\hat{\underline{\lambda}}}{Dt} = \cancelto{0}{\frac{\partial\hat{\underline{\lambda}}}{\partial t}} + \frac{D\lambda}{Dt}\frac{\partial\hat{\underline{\lambda}}}{\partial\lambda} + \frac{D\phi}{Dt}\frac{\partial\hat{\underline{\lambda}}}{\partial\phi} + \frac{Dr}{Dt}\frac{\partial\hat{\underline{\lambda}}}{\partial r}$$

$$= \frac{u}{r\cos\phi}\frac{\partial\hat{\underline{\lambda}}}{\partial\lambda} + \frac{v}{r}\frac{\partial\hat{\underline{\lambda}}}{\partial\phi} + w\frac{\partial\hat{\underline{\lambda}}}{\partial r}$$

Seek expression for $\dfrac{\partial\hat{\underline{\lambda}}}{\partial\lambda}$

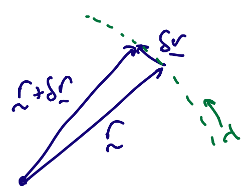
How do we describe the unit vectors $\hat{\underline{\lambda}}, \hat{\underline{\phi}}, \hat{\underline{r}}$

$\hat{\underline{\lambda}}$ is a unit vector in the direction in which λ is increasing.

Define $\underline{r}$ as the position vector. May express as

$$\underline{r} = \underline{r}(\lambda, \phi, r)$$

Consider $$\delta\underline{r} = \underline{r}(\lambda+\delta\lambda, \phi, r) - \underline{r}(\lambda, \phi, r)$$



In the limit of $\delta\lambda \to 0$, we have

$\delta\underline{r} = \dfrac{\partial\underline{r}}{\partial\lambda}\delta\lambda$ is a vector in the direction of increasing λ.

Hence $\quad \hat{\underset{\sim}{\lambda}} = \dfrac{\delta \underset{\sim}{r}}{|\delta \underset{\sim}{r}|} = \dfrac{\frac{\partial \underset{\sim}{r}}{\partial \lambda}}{\left|\frac{\partial \underset{\sim}{r}}{\partial \lambda}\right|}$ $\quad$ (Divide by $\left|\frac{\partial r}{\partial \lambda}\right|$ to make length $=1$)

To be useful, need to express $\hat{\underset{\sim}{\lambda}}$ in a set of cartesian co-ordinates.

Remember $\quad$ $x = r\cos\lambda\cos\phi$
$y = r\sin\lambda\cos\phi$
$z = r\sin\phi$

In cartesian coords:

$$\underset{\sim}{r} = x\hat{\underset{\sim}{x}} + y\hat{\underset{\sim}{y}} + z\hat{\underset{\sim}{z}}$$

$$\frac{\partial \underset{\sim}{r}}{\partial \lambda} = \frac{\partial x}{\partial \lambda}\hat{\underset{\sim}{x}} + \frac{\partial y}{\partial \lambda}\hat{\underset{\sim}{y}} + \frac{\partial z}{\partial \lambda}\hat{\underset{\sim}{z}}$$

$$= -r\sin\lambda\cos\phi\,\hat{\underset{\sim}{x}} + r\cos\lambda\cos\phi\,\hat{\underset{\sim}{y}}$$

$$\left|\frac{\partial \underset{\sim}{r}}{\partial \lambda}\right| = \left(r^2\sin^2\lambda\cos^2\phi + r^2\cos^2\lambda\cos^2\phi\right)^{\frac{1}{2}}$$

$$= r\cos\phi$$

$$\hat{\underset{\sim}{\lambda}} = -\sin\lambda\,\hat{\underset{\sim}{x}} + \cos\lambda\,\hat{\underset{\sim}{y}} \qquad \cdots (A)$$

Can also show that

$$\hat{\underset{\sim}{\phi}} = -\cos\lambda\sin\phi\,\hat{\underset{\sim}{x}} - \sin\lambda\sin\phi\,\hat{\underset{\sim}{y}} + \cos\phi\,\hat{\underset{\sim}{z}} \qquad \cdots (B)$$

$$\hat{\underset{\sim}{r}} = \cos\lambda\cos\phi\,\hat{\underset{\sim}{x}} + \sin\lambda\cos\phi\,\hat{\underset{\sim}{y}} + \sin\phi\,\hat{\underset{\sim}{z}} \qquad \cdots (C)$$

Returning to our expression

$$\frac{D\hat{\lambda}}{Dt} = \frac{u}{r\cos\phi}\frac{\partial\hat{\lambda}}{\partial\lambda} + \frac{v}{r}\cancel{\frac{\partial\hat{\lambda}}{\partial\phi}} + w\cancel{\frac{\partial\hat{\lambda}}{\partial r}}$$

0 by (A) $\qquad$ 0 by (A)

By (A):
$$\frac{\partial\hat{\lambda}}{\partial\lambda} = -\cos\lambda\,\hat{x} - \sin\lambda\,\hat{y}$$

Need to express this in terms of $(\hat{\lambda}, \hat{\phi}, \hat{r})$

General solution: invert equations (A), (B) & (C)

In this case, can see that $\frac{\partial\hat{\lambda}}{\partial\lambda}$ points inwards towards the z axis. In particular, has no component in $\hat{\lambda}$

From (B) and (C) see that

$$\begin{aligned}\sin\phi\,\hat{\phi} - \cos\phi\,\hat{r} &= (-\cos\lambda\sin^2\phi - \cos\lambda\cos^2\phi)\,\hat{x} \\ &\quad (-\sin\lambda\sin^2\phi - \sin\lambda\cos^2\phi)\,\hat{y} \\ &\quad (\sin\phi\cos\phi - \sin\phi\cos\phi)\,\hat{z} \\ &= -\cos\lambda\,\hat{x} - \sin\lambda\,\hat{y} \\ &= \frac{\partial\hat{\lambda}}{\partial\lambda}\end{aligned}$$

$$\begin{aligned}\frac{D\hat{\lambda}}{Dt} &= \frac{u}{r\cos\phi}\left(-\sin\phi\,\hat{\phi} - \cos\phi\,\hat{r}\right) \\ &= -\frac{u\tan\phi}{r}\hat{\phi} - \frac{u}{r}\hat{r}\end{aligned}$$

$$\frac{D(u\hat{\lambda})}{Dt} = \frac{Du}{Dt}\hat{\lambda} - \frac{u^2\tan\phi}{r}\hat{\phi} - \frac{u^2}{r}\hat{r}$$

Do the same for $v\hat{\phi}$ and $w\hat{r}$ to get full sol^{n}.

Problem 3.

(a) Thermodynamic eq^{n}

$$c_v \frac{DT}{Dt} + p\frac{D\alpha}{Dt} = Q$$

Adiabatic flow: $Q=0$

$$c_v \frac{DT}{Dt} + p\frac{D\alpha}{Dt} = 0 \quad \cdots (1)$$

now, $p\alpha = R_d T$

$$\frac{D(p\alpha)}{Dt} = R_d \frac{DT}{Dt}$$

$$\alpha \frac{Dp}{Dt} + p\frac{D\alpha}{Dt} = R_d \frac{DT}{Dt}$$

$$p\frac{D\alpha}{Dt} = R_d \frac{DT}{Dt} - \alpha\frac{Dp}{Dt}$$

Substitute into (1):

$$c_v \frac{DT}{Dt} + R_d \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = 0$$

let $c_p = c_v + R_d$

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = 0$$

Divide by T $$\frac{c_p}{T}\frac{DT}{Dt} - \frac{\alpha}{T}\frac{Dp}{Dt} = 0$$

$$p\alpha = R_d T$$

$$\frac{\alpha}{T} = \frac{R_d}{p}$$

$$\frac{c_p}{T}\frac{DT}{Dt} - \frac{R_d}{p}\frac{Dp}{Dt} = 0$$

Divide by c_p: $$\frac{1}{T}\frac{DT}{Dt} - \frac{R_d}{c_p}\left(\frac{1}{p}\frac{Dp}{Dt}\right) = 0$$

$$\Rightarrow \quad \frac{D}{Dt}\left\{ \ln(T) - \frac{R_d}{c_p}\ln(p) \right\} = 0$$

Add $\quad \frac{D}{Dt}\left(\frac{R_d}{c_p}\ln(p_0)\right) = 0$

$$\frac{D}{Dt}\left\{ \ln(T) - \frac{R_d}{c_p}\ln(p) + \frac{R_d}{c_p}\ln(p_0) \right\} = 0$$

$$\frac{D}{Dt}\left\{ \ln(T) - \ln\left[\left(\frac{p}{p_0}\right)^{\frac{R_d}{c_p}}\right] \right\} = 0$$

$$\frac{D}{Dt}\left\{ \ln\left[T\left(\frac{p_0}{p}\right)^{\frac{R_d}{c_p}}\right] \right\} = 0$$

$$\frac{D}{Dt}\left(T\left(\frac{p_0}{p}\right)^{\frac{R_d}{c_p}}\right) = 0$$

$$\frac{D\theta}{Dt} = 0.$$

(b) Buoyancy $\quad b = -g\frac{(\rho - \rho_0)}{\rho} \quad$ ρ = parcel density, ρ_0 = environment density

Assume parcel & environment have same pressure

have then $\quad \rho = \frac{p}{R_d \theta \pi} \quad$ & $\quad \rho_0 = \frac{p}{R_d \theta_0 \pi}$

$$b = -g\,\frac{\left(\frac{p}{R_d\theta\pi} - \frac{p}{R_d\theta_0\pi}\right)}{\frac{p}{R_d\theta\pi}} = -g\,\frac{\left(\frac{1}{\theta} - \frac{1}{\theta_0}\right)}{1/\theta}$$

Multiply top & bottom by $\theta\theta_0$

$$b = -g\left(\frac{\theta_0 - \theta}{\theta_0}\right) = g\frac{(\theta - \theta_0)}{\theta_0}$$

Now, suppose we have an environment with

$$\theta_0 = \theta_0(z)$$

Perturb a parcel within this environment from $z \to z + \delta z$

Parcel conserves θ, so at $z + \delta z$

parcel: $\quad \theta = \theta_0(z)$

environment: $\quad \theta_0 = \theta_0(z + \delta z)$

Hence

$$b = \left(\frac{\theta_0(z) - \theta_0(z+\delta z)}{\theta_0(z+\delta z)} \right) g$$

$$b = \frac{-g}{\theta_0(z+\delta z)} \, \delta z \left(\frac{\theta_0(z+\delta z) - \theta_0(z)}{\delta z} \right)$$

for $\delta z \to 0$

$$b = \frac{-g}{\theta_0} \frac{\partial \theta_0}{\partial z} \delta z$$

(c) Have $\dfrac{Dw}{Dt} = b$

$$\frac{Dw}{Dt} = \frac{-g}{\theta_0} \frac{\partial \theta_0}{\partial z} \delta z$$

vertical displacement of parcel.

$$w = \frac{D\delta z}{Dt}$$

$$\frac{D^2 \delta z}{Dt^2} = -\frac{g}{\theta_0} \frac{\partial \theta_0}{\partial z} \delta z$$

$$\frac{D^2 \delta z}{Dt^2} = -N^2 \delta z$$

Brunt Väisälä freq.

$$\delta z(t) = A \exp(iNt) + B \exp(-iNt)$$

for $\frac{\partial\theta_0}{\partial z} > 0$, $N^2 > 0$: oscillation, freq of N.

for $\frac{\partial\theta_0}{\partial z} < 0$, $N^2 < 0$: exponential growth: unstability.

Problem 4.

Assume no pressure gradients:

$$\frac{Du}{Dt} = fv \qquad \cdots(1)$$

$$\frac{Dv}{Dt} = -fu \qquad \cdots(2)$$

$\frac{D}{Dt}$ of Eq (1):

$$\frac{D^2u}{Dt^2} = f\frac{Dv}{Dt} \qquad (f\text{-plane})$$

Substitute into (2)

$$\frac{D^2u}{Dt^2} = -f^2u$$

Initial cond: $u = u_0$

$v = \frac{1}{f}\frac{Du}{Dt} = 0$

Solution is

$$u = A\cos(ft) + B\sin(ft)$$

Initial condition gives $A = u_0$, $B = 0$

$$u = u_0\cos(ft)$$

$$v = \frac{1}{f}\frac{Du}{Dt} = -u_0\sin(ft)$$

Remember $u = \frac{Dx}{Dt} \Rightarrow x = x_0 + f u_0 \sin(ft)$

$v = \frac{Dy}{Dt} \Rightarrow y = y_0 + f u_0 \cos(ft)$

Circular motion with frequency $\frac{f}{2\pi} = \frac{2\Omega \sin\phi}{2\pi}$

- These are called inertial circles.
- Can be seen in the ocean
- Frequency highest at poles, goes to zero at equator