Advanced Dynamical Meteorology

Lecture 2

Waves

What is a wave? What do you think a wave is?

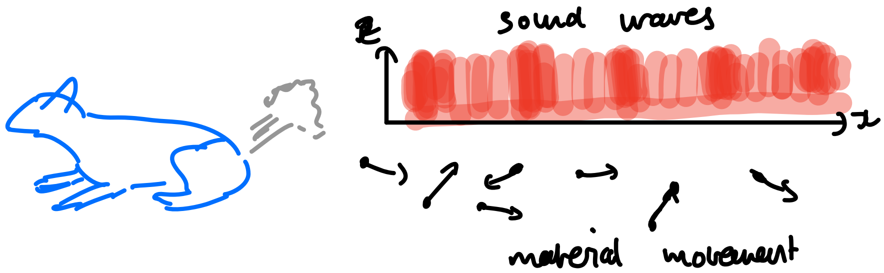
Hard to define rigorously, but has some key characteristics

1) Propagation

Waves travel by propagation

This means they don't transport material, but they can transport other things, like energy or information

A dog passes gas. You hear it, but it takes a few seconds before you smell it



This is because sound propagates through waves.

But to smell something, molecules of gas must enter your nose

How might this be relevant in the atmosphere?

2) Dispersion relation

Have you heard of this?

Waves in general have a relationship between their "size" and their frequency.

Linear waves (the type of waves we will consider here) have a relationship between their frequency and wavenumber called the dispersion relation.

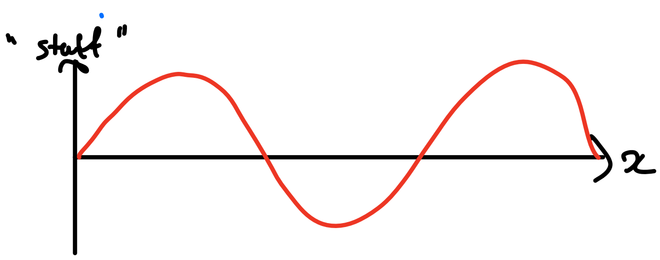
$$\underset{\text{frequency}}{\omega} = \Omega(\underset{\text{wavenumber}}{k})$$

$$k = \frac{2\pi}{\lambda} \leftarrow \text{wavelength}$$

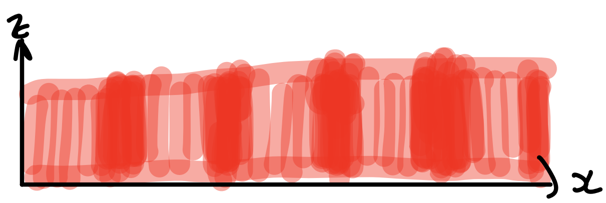
The dispersion relation is super important! We will see why soon.

3) Oscillations??

When we think of waves, we generally think of "sin" and "cos" functions:



transverse wave



longitudinal wave

But remember Fourier's Theorem

Any function can be expressed as the sum of sines and cosines:

$$\psi(x) = \int_{-\infty}^{\infty} A(k)\cos(kx) + B(k)\sin(kx)\ dk$$

So if you sum up enough of them, you can make any shape!

<u>But</u>: Remember that linear waves have a relationship between their frequency and wavenumber

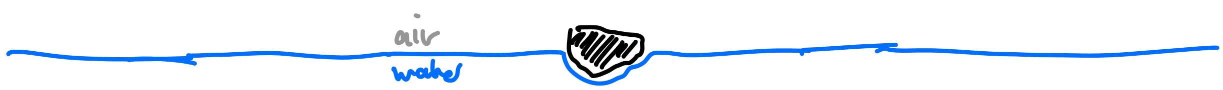
This also tells us about the "speed" of a wave.

If the speed of a wave varies with wavelength, the waves will eventually separate by wavelength

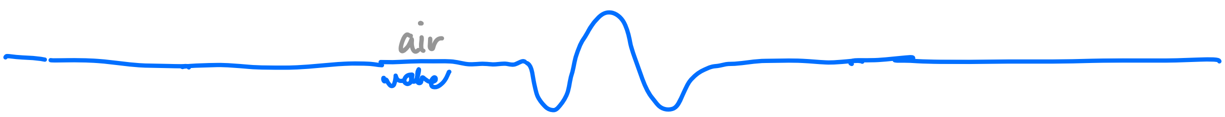
Such a wave is called <u>dispersive</u>

Far away from a wave source, the waves tend to look like plane waves.

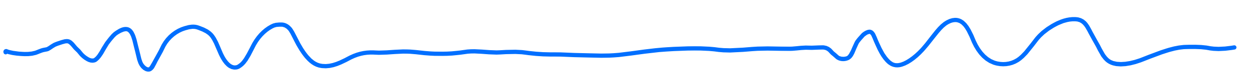
Example: Drop a rock in a pond:

1) 

Rock makes irregular depression in water surface

2) 

Water rebounds and creates a wave

3) 

Ripples propagate away like plane waves

This is also why beach waves are so regular and why they come in sets.

Analysing waves

Lets make this a bit more concrete. How do we actually analyse waves?

There exists a wide literature of mathematical theory on this.

Here we will introduce some heuristics to help understand waves

Can be confusing, and terms are often not explained.

Hopefully this lecture will help.

Suppose we have a governing equation for a physical system of interest. Often we can combine set of PDEs into single equation. We will see this for the Quasigeostrophic equations later.

Equation may be written in general form:

$$f\left(\psi, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial t}, \frac{\partial^2 \psi}{\partial x^2}, \frac{\partial^2 \psi}{\partial t^2}, \dots\right) = 0$$

for $\psi = \psi(x,t)$ and some function f.

While non-linear waves exist, usually in meteorology, we will study linear waves.

This means we need to linearise the equation

Linearisation: split solution into background state and small perturbations around that background state

E.g., small perturbations around mean jet

Lets take a simple example of a governing equation:

$$\frac{\partial^2 \varphi}{\partial t^2} - c^2 \frac{\partial^2 \varphi}{\partial x^2} - \kappa \frac{\partial \varphi}{\partial x} - \kappa_2 \left(\frac{\partial \varphi}{\partial x}\right)^2 = 0$$

Lets linearise by supposing

$$\varphi(x,t) = \overline{\varphi} + \varphi'(x,t), \qquad |\varphi'| << \overline{\varphi}$$

We then have $\quad \varphi'_{tt} + c^2 \varphi'_{xx} - \kappa \varphi'_x - \cancel{\kappa_2 (\varphi_x)^2} = 0$

Linearisation means we <u>neglect products of perturbed quantities</u>

Linearised equation:

$$\varphi'_{tt} - c^2 \varphi'_{xx} - \kappa \varphi'_x = 0$$

or

$$\left[\partial_{tt} - c^2 \partial_{xx} - \kappa \partial_x\right] \varphi' = 0$$

or

$$\mathcal{L}(\varphi) = 0$$

where $\mathcal{L}$ is a linear operator.

Note that our equation has constant co-efficients.
This does not have to be true.

Okay, we have the linear equation. Now what?

We look for "wave-like" solutions:

$$\psi'(x,t) = \mathrm{Re}\left\{ A e^{i(kx-\omega t)} \right\}$$

This is an oscillating solution with wavenumber k and angular frequency ω.

Remember $e^{i\theta} = \cos\theta + i\sin\theta$

We can assume a solution of cosine or sine, but using the exponential form is much easier.

In general the amplitude A is complex.

Let's apply this to our simple linear equation:

$$\left[\partial_{tt} - c^2\partial_{xx} - \kappa\partial_x\right]\psi' = 0$$

$$\partial_t\psi' = -i\omega A e^{i(kx-\omega t)} = -i\omega\psi'$$

$$\partial_x\psi' = i k A e^{i(kx-\omega t)} = ik\psi'$$

The equation becomes:

$$\left[(-i\omega)^2 - c^2(ik)^2 - \kappa ik\right]\psi' = 0$$

Either $\psi' = 0$ [trivial solution]

or

$$\omega^2 = c^2k^2 - i\kappa k$$

This is the dispersion relation!

Let's consider first the case $\kappa = 0$. In this case the dispersion relation is simple:

$$\omega = \pm ck$$

Describing two waves propagating left and right.

This is actually a very special equation called the "wave equation"

It admits two wave solutions (+ and −), which are non-dispersive. We will discuss what this means below.

Note: The "wave-like" solutions we have found need not be the only solutions. There can be other so-called "non-normal mode" solutions that do not have this structure.

Nonetheless, we can learn a lot from these exponential solutions.

How does a wave propagate?
What conditions do you need for a wave to propagate?

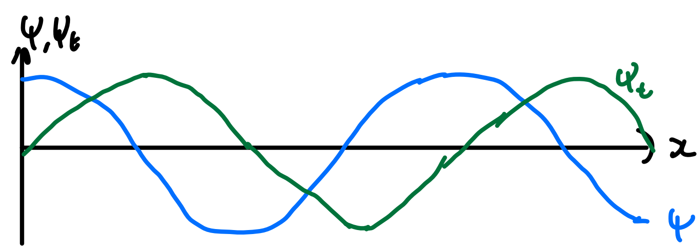
Wave propagation

Let's consider the wave-like solutions we have found.
To be concrete, suppose solution is a cosine:

$$\psi(x,t) = A\cos(kx - \omega t) \qquad \text{(dropping the prime)}$$

for some $\omega = ck$. (ω & k are real)

We also have $\psi_t(x,t) = +A\omega \sin(kx - \omega t)$



φ is in quadrature with φ_t

φ_t is shifted to the right from φ

$\Rightarrow$ wave propagates to the right

When $\varphi = 0$ $|\varphi_t|$ is a maximum

$\Rightarrow$ wave does not amplify or decay.

This is the fundamental mechanism of wave propagation!

Let's go back to the non-zero K case. In that case, we have,

$$\varphi(x,t) = \mathrm{Re}\left\{A e^{i(kx-\omega t)}\right\}$$

with:

$$\omega^2 = c^2k^2 - iKk$$

$$\omega = (c^2k^2 - iKk)^{\frac{1}{2}}$$

$$= ck\left(1 - iK\frac{k}{c^2k^2}\right)^{\frac{1}{2}}$$

Using a Tayor series:

$$(1+x)^{\frac{1}{2}} = 1 + \left.\frac{dx^{\frac{1}{2}}}{dx}\right|_{x=1} x + O(x^2)$$

$$= 1 + \frac{x}{2} + O(x^2)$$

$$\omega \simeq ck - \frac{iK}{2c}$$

$$= \omega_r - \frac{iK}{2c}$$

We may then write our solution:

$$\psi(x,t) = \mathrm{Re}\left\{ A e^{i(kx-\omega t)} \right\}$$

$$= \mathrm{Re}\left\{ A e^{i(kx-\omega_r t)} \cdot e^{i^2\left(\frac{\kappa t}{2c}\right)} \right\}$$

$$= \mathrm{Re}\left\{ A e^{-\frac{\kappa t}{2c}} e^{i(kx-\omega_r t)} \right\}$$

If we take A to be real, we have:

$$\psi(x,t) = A e^{-\frac{\kappa t}{2c}} \cos(kx - \omega_r t)$$

The wave is damped! Its amplitude decreases with time.

This corresponds to a negative imaginary component of ω:

$$\mathrm{Im}\{\omega\} < 0$$

We can also calculate $\psi_t(x,t)$:

$$\psi_t(x,t) = e^{-\kappa t/2c} \frac{\partial}{\partial t} A\cos(kx - \omega_r t)$$

$$+ A\cos(kx-\omega_r t) \frac{\partial}{\partial t}\left(e^{-\kappa t/2c}\right)$$

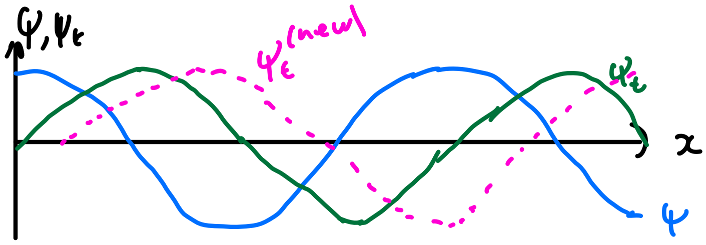
$$= Ae^{-\kappa t/2c}\left(\omega_r \sin(kx-\omega_r t)\right)$$

$$- Ae^{-\frac{\kappa t}{2c}} \cdot \frac{\kappa}{2c} \cos(kx-\omega_r t)$$

$$= Ae^{-\frac{\kappa t}{2c}} \left\{ \underbrace{-\frac{\kappa}{2c}\cos(kx-\omega_r t)}_{\text{in phase}} + \underbrace{\omega_r \sin(kx-\omega_r t)}_{\text{in quadrature}} \right\}$$

The rate of change now has a component opposing the amplitude.
This is damping.

This shifts ψ_t relative to ψ.



Summary:

$$\omega = \Omega(k)$$

$$= \omega_r + i\omega_i$$

speed of propagation (ω_r)   damping/amplification (ω_i)

Under what conditions would ω_i be positive?

Phase speed and Group velocity

Waves propagate. But what speed do they propagate at?
This is a surprisingly complicated question.

There are two different speeds we might be interested in.

Phase speed:

$$c_p = \frac{\omega}{k}$$

This is the speed at which individual troughs and ridges move.

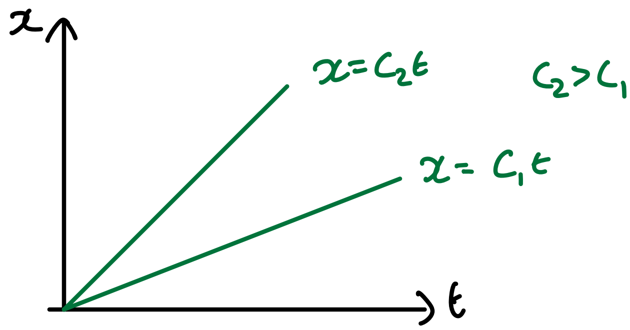
We can see this by following an individual wavecrest.

Suppose $\psi(x,t) = A\cos(kx-\omega t)$

$$= A\cos\left(k[x-c_p t]\right)$$

wave crests occur when: $x - c_p t = 0$

so they follow a path: $x = c_p t$



Phase speed can be defined for each direction:

For a 2-D wave

$$\psi(x,y,t) = A e^{i(kx+ly-\omega t)}$$

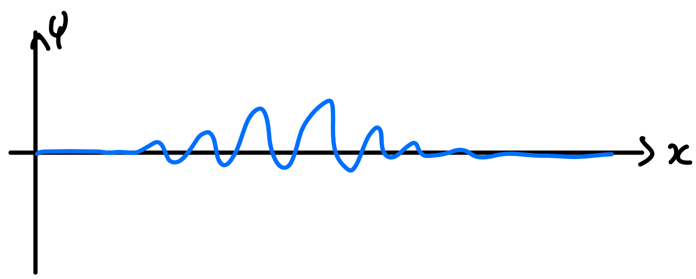
$$\omega = \Omega(k,l)$$

$$c_{px} = \frac{\omega}{k}, \quad c_{py} = \frac{\omega}{l}$$

Group velocity

For plane waves, phase speed is all we need.
But in general, waves are made up of a range of frequencies.

E.g., suppose we have a wave packet:



We might describe this as a single wave with an amplitude modulation:

$$\psi(x,t) = \underbrace{Ae^{-\left(\frac{x}{L}\right)^2}}_{\text{Gaussian envelope}} e^{i(kx-\omega t)}$$

Can be expressed as sum of plane waves:

$$\psi(x,t) = \int_{-\infty}^{\infty} A\, e^{-\left(\frac{k'-k}{\Delta k}\right)^2} e^{i(k'x-\omega t)}\, dk'$$

where $\Delta k \propto \frac{1}{L}$ and $\omega = \Omega(k)$

This is because the Fourier transform of a Gaussian is a Gaussian.

For simplicity, consider sum of two waves:

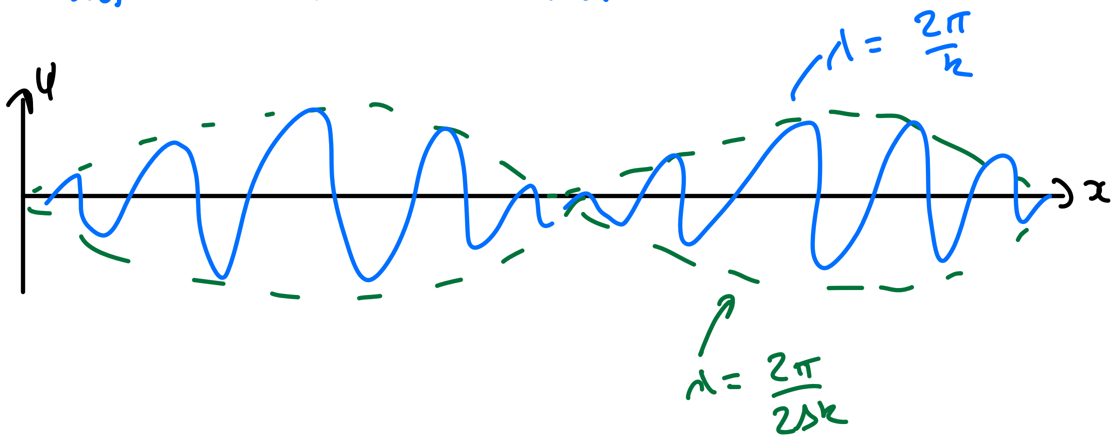
$$\psi(x,t) = Ae^{i[(k+\Delta k)x - (\omega+\Delta\omega)t]} + Ae^{i[(k-\Delta k)x-(\omega-\Delta\omega)t]}$$

$$= Ae^{i(kx-\omega t)}e^{i(\Delta kx - \Delta\omega t)} + Ae^{i(kx-\omega t)}e^{-i(\Delta kx-\Delta\omega t)}$$

$$= A\left[e^{i(\Delta kx-\Delta\omega t)} + e^{-i(\Delta kx-\Delta\omega t)}\right]e^{i(kx-\omega t)}$$

$$= A\cos(\Delta kx - \Delta\omega t)\, e^{i(kx-\omega t)}$$

This looks like a "Beat":



We know the speed at which the blue crests move is:

$$c_p = \frac{\omega}{k}$$

What speed does the green envelope move at?

This is governed by

$$Env = A\cos(\Delta k x - \Delta\omega t)$$

$$= A\cos\left[\Delta k\left(x - \frac{\Delta\omega}{\Delta k}t\right)\right]$$

The speed of a group of waves is:

$$c_g = \frac{\Delta\omega}{\Delta k}$$

For $\Delta\omega \to 0$ this is

$$c_g = \frac{\partial\omega}{\partial k}$$

This is the <u>group velocity</u>

It is the velocity that transports energy and information.

For our simple example

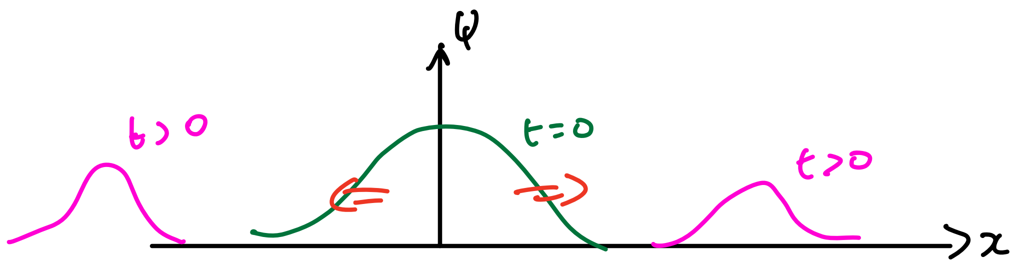
$$\omega = ck$$

$$\frac{\partial \omega}{\partial k} = c$$

The phase speed and group velocities are the same!

This is what is meant by non-dispersive.

$\Rightarrow$ Disturbances propagate without dispersion:



But most waves are dispersive

$$c_p \neq c_g$$

$\Rightarrow$ Energy travels at different speed to phase.

$\Rightarrow$ Waves separate by wavenumber away from their source.

<u>Inhomogenous media</u>

Everything we have discussed assumed the governing equation had constant co-efficients.

We then know that an x-derivative gives ikx and a time derivative gives $-i\omega t$.

This allows us to find the dispersion relation.

But what happens if the co-efficients vary in space?

Two possible approaches:

1) Don't use a wave-like form in the direction of rotation

Suppose the equation varies in z, but the coefficients are constant in x and y.

We can search for <u>separable</u> solutions of the form

$$\psi(x,y,z,t) = A(z)e^{i(kx+ly-\omega t)}$$

We can then reduce the problem to an ODE in z. Usually this results in an eigenvalue problem for discrete vertical structures.

This method used when boundaries are involved and/or the variation occurs over a similar length scale as the wavelength of interest.

2) If we can assume the variation occurs on a length scale large compared to the wavelength, we can apply <u>ray theory</u>.

Assume a wave of the form:

$$\psi(x,t) = A(x)\,e^{i(kx-\omega t)}$$

where A is a slowly-varying amplitude, and the dispersion relation is satisfied locally:

$$\omega = \Omega(k;x)$$

Large body of work on ray theory, and related method of WKB. Here we sketch some main results.

→ Energy is propagated by the group velocity. We therefore follow characteristics of the group velocity as rays.

→ If equation is not an explicit function of t, ω is constant along rays.

→ If equation is not an explicit function of x, k is constant along rays.

→ If both above conditions satisfied, can use dispersion relation to find $l(y)$ as wave propagates meridionally.

Critical Lines & Turning lines

Consider a barotropic Rossby wave in a background flow $U(y)$

As we will show later the dispersion relation for such a wave is:

$$\omega = U(y)k - \frac{\beta k}{k^2 + l^2}$$

Here, β is the gradient of absolute vorticity,

$$\beta = \frac{\partial}{\partial y}(f + \zeta)$$

$$= \frac{\partial}{\partial y}\left(f - \frac{\partial U(y)}{\partial y}\right)$$

Since coefficients in equation are independent of (x, t), we know (k, ω) are constant along rays.

We can find how l varies along a ray by rearranging:

$$u - c_{px} = \frac{\beta}{k^2+l^2} \quad , \quad c_{px} = \frac{\omega}{k}$$

$$k^2+l^2 = \frac{\beta}{u - c_{px}}$$

$$l^2 = \frac{\beta}{u - c_{px}} - k^2$$

To make things simple, assume k is small (waves are long)

$$l^2 = \frac{\beta}{u - c_{px}}$$

For propagation, need $l^2 > 0$

otherwise l is imaginary. What does it mean if $l^2 < 0$?

This requires $\beta > 0$. But suppose $\beta \to 0$, what happens to the wave?

In this case, $l \to 0$, the wave is reflected.

$\to c_{gy} \to 0$, $c_{gx} > 0$

$\to$ The waves become long, and don't break.

$\to$ The line $l = 0$ is a <u>turning line</u>

Alternatively, when $u - c_{px} \to 0$, $l \to \infty$

This is called a critical line. Here the wavelength shortens, becomes non-linear and breaks.

At a critical line:

$\Rightarrow$ $l \to \infty$

$\Rightarrow$ $c_{gx} = c_{gy} = 0$

$\Rightarrow$ wave breaks and is absorbed

Ray theory, with turning and critical lines, is a powerful way of analysing wave problems without solving complicated PDEs.

Summary

⇒ A wave is characterised by its dispersion relation

$$\omega = \Omega(k, l, m)$$

⇒ Dispersion relation tells us

- the phase speed $c_{px} = \frac{\omega}{k}$, $c_{py} = \frac{\omega}{l}$

- the group velocity $c_{gx} = \frac{\partial \omega}{\partial k}$, $c_{gy} = \frac{\partial \omega}{\partial l}$

⇒ The group velocity determines the energy propagation of the wave and the propagation of the envelope.

⇒ Real part of $\omega = \Omega(k)$ corresponds to wave propagation, imaginary part to growth/decay.

⇒ Can apply these results to "slowly" varying media.

This is what we will now do!