Lecture 7 : The angular momentum budget

Zonal-mean midlatitude circulation

Characterised by:

[1] - Strong upper level westerly jets

[2] - strong latitudinal temperature gradient

[3] - westerly surface winds

[4] - thermally indirect overturning

[5] - Strong EKE maximum

Why?

Thermal wind balance

Relates points [1] and [2]

As we have seen, combination of geostrophic & hydrostatic balance gives

$$2\Omega \sin\phi \frac{\partial u}{\partial p} = \frac{1}{R_e}\left(\frac{R_d}{p}\right)\frac{\partial T}{\partial \phi}$$

or in more familiar notation:

$$\frac{\partial u}{\partial p} = \frac{R_d}{pf}\frac{\partial T}{\partial y}$$

- Relates vertical shear to horizontal temp. gradients
- Explains upper level winds, but not surf. winds

What about points [2-5]?

Consider the angular-momentum budget.

Angular momentum

Recall the definition of angular momentum

$$M = (\Omega R_e \cos\phi + u) R_e \cos\phi$$

$$= \Omega R_e^2 \cos^2\phi + u R_e \cos\phi$$

planetary angular momentum — relative angular momentum

Also recall that M is conserved for axisymmetric, frictionless flow.

More generally we have

$$\frac{DM}{Dt} = -\frac{\partial \bar{\sigma}}{\partial \lambda} + R_e \cos\phi \, F_\lambda$$

Divide angular momentum into planetary and relative components

$$M_p = \Omega R_e^2 \cos^2\phi \quad , \qquad M_r = u R_e \cos\phi$$

Consider planetary component:

$$M_p = M_p(\phi)$$

Hence
$$\frac{DM_p}{Dt} = \cancel{\frac{\partial M_p}{\partial t}} + \cancel{\frac{u}{R_e\cos\phi}\frac{\partial M_p}{\partial \lambda}} + \frac{v}{R_e}\frac{\partial M_p}{\partial \phi} + \cancel{\omega\frac{\partial M_p}{\partial p}}$$

$$= v\frac{\partial M_p}{\partial \phi}$$

$$\Rightarrow \quad \frac{DM_p}{Dt} = \frac{v}{R_e}\frac{\partial M_p}{\partial \phi} = -v\,\Omega R_e\,(2\cos\phi\sin\phi)$$

$$= -(2\Omega\sin\phi)\, v\, (R_e\cos\phi)$$

$$= -R_e\cos\phi\,(fv)$$

Now, $$\frac{DM}{Dt} = \frac{DM_r}{Dt} + \frac{DM_p}{Dt} = -\frac{\partial \Phi}{\partial \lambda} + R_e\cos\phi\,(F_\lambda)$$

$$\Rightarrow \quad \frac{DM_r}{Dt} - R_e\cos\phi(fv) = -\frac{\partial \Phi}{\partial \lambda} + R_e\cos\phi(F_\lambda)$$

So that, $$\frac{DM_r}{Dt} = -\frac{\partial \Phi}{\partial \lambda} + R_e\cos\phi\left\{fv + F_\lambda\right\}$$

$$\frac{\partial M_r}{\partial t} + \underline{u}\cdot\nabla M_r = -\frac{\partial \Phi}{\partial \lambda} + R_e\cos\phi\,(fv + F_\lambda) \qquad \cdots(1)$$

Flux form

Remember, can use continuity to write (1) in flux form

For relative angular momentum, we have

$$\frac{\partial M_r}{\partial t} + \nabla\cdot(\underline{u}M_r) = -\frac{\partial \Phi}{\partial \lambda} + R_e\cos\phi\left\{fv + F_\lambda\right\}$$

Upper tropospheric angular-momentum budget:

In upper troposphere, friction is weak $\Rightarrow$ $F_\lambda = 0$

Take the zonal & time mean to give:

$$\cancelto{0}{\frac{\partial}{\partial t}[\overline{M_r}]} + \nabla\cdot\left([\overline{\underline{u}M_r}]\right) = \cancelto{0}{-\frac{\partial \Phi}{\partial \lambda}} + R_e\cos\phi\, f[\overline{v}]$$

assuming no long term trend in M_r

by periodicity

Rearranging,

$$f[\overline{v}] = \frac{1}{R_e\cos\phi}\underbrace{\left\{\frac{1}{R_e\cos\phi}\frac{\partial}{\partial\phi}\left([\overline{vM_r}]\cos\phi\right) + \frac{\partial}{\partial p}[\overline{\omega M_r}]\right\}}_{\text{zonal-mean divergence in spherical coords}}$$

Now, since $M_r = R_e \cos\varphi u$, we have

$$f[\bar{v}] = \frac{1}{R\cos^2\varphi} \frac{\partial}{\partial \varphi}\left([\overline{uv}]\cos^2\varphi\right) + \frac{\partial}{\partial p}\left([\overline{u\omega}]\right)$$

This is an equation for the meridional flow in the upper troposphere!

- → RHS is divergence of flux of (angular) momentum
- → Flux is dominated by horizontal component
- → Can be divided into mean & eddy
- → In midlatitudes, dominated by transient eddies

Divide into mean & eddy:

$$f[\bar{v}] = \frac{1}{R_e\cos^2\varphi} \frac{\partial}{\partial\varphi} [\bar{u}][\bar{v}]\cos^2\varphi + \frac{\partial}{\partial p}[\bar{u}][\bar{\omega}] \quad \text{mean}$$

$$+ \frac{1}{R_e\cos^2\varphi} \frac{\partial}{\partial\varphi}[\bar{u}^*\bar{v}^*]\cos^2\varphi + \frac{\partial}{\partial p}[\bar{u}^*\bar{v}^*] \quad \text{stationary}$$

$$+ \frac{1}{R_e\cos^2\varphi} \frac{\partial}{\partial\varphi}[\overline{u'v'}] + \frac{\partial}{\partial p}[\overline{u'\omega'}] \quad \text{transient}$$

Show Figure!

Another useful version can be derived by rearranging the mean transport term:

$$T_{mean} = \frac{1}{R_e \cos^2\phi}\frac{\partial}{\partial \phi}\left([\bar{u}][\bar{v}]\cos^2\phi \right) + \frac{\partial}{\partial p}\left([\bar{u}][\bar{\omega}] \right)$$

$$\frac{\partial(\alpha\beta)}{\partial\phi} = \alpha\frac{\partial\beta}{\partial\phi} + \beta\frac{\partial\alpha}{\partial\phi}$$

$$\alpha = [\bar{u}]\cos\phi$$
$$\beta = [\bar{\rho}][\bar{v}]\cos\phi$$

$$= \frac{[\bar{v}]}{R_e\cos\phi}\frac{\partial}{\partial\phi}[\bar{u}]\cos\phi + \cancelto{0}{\frac{[\bar{u}]}{R_e\cos\phi}\frac{\partial}{\partial\phi}\left([\bar{v}]\cos\phi\right)}$$
$$+ [\bar{\omega}]\frac{\partial[\bar{u}]}{\partial p} + \cancel{[\bar{u}]\frac{\partial[\bar{\omega}]}{\partial p}}$$

continuity

$$\alpha = [\bar{u}]$$
$$\beta = [\bar{\rho}][\bar{\omega}]$$

The remaining terms correspond to angular momentum advection; they are dominated by the horizontal component.

$$T_{mean} \approx \frac{[\bar{v}]}{R_e\cos\phi}\frac{\partial}{\partial\phi}\left(\bar{u}\cos\phi\right) = -[\bar{v}][\bar{\zeta}]$$

ζ is the vertical component of vorticity:

$$\zeta = (\nabla \times \underset{\sim}{u}) \cdot \hat{k}$$

$$\zeta = \frac{1}{R_e\cos\phi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial\, u\cos\phi}{\partial\phi}\right]$$

$$[\bar{\zeta}] = \frac{-1}{R_e\cos\phi}\frac{\partial}{\partial\phi}\left([\bar{u}]\cos\phi\right)$$

Thus we may write the steady-state angular momentum budget,

$$[\bar{v}]\left\{ f + [\bar{\zeta}] \right\} = -S$$

where S is the eddy momentum flux convergence

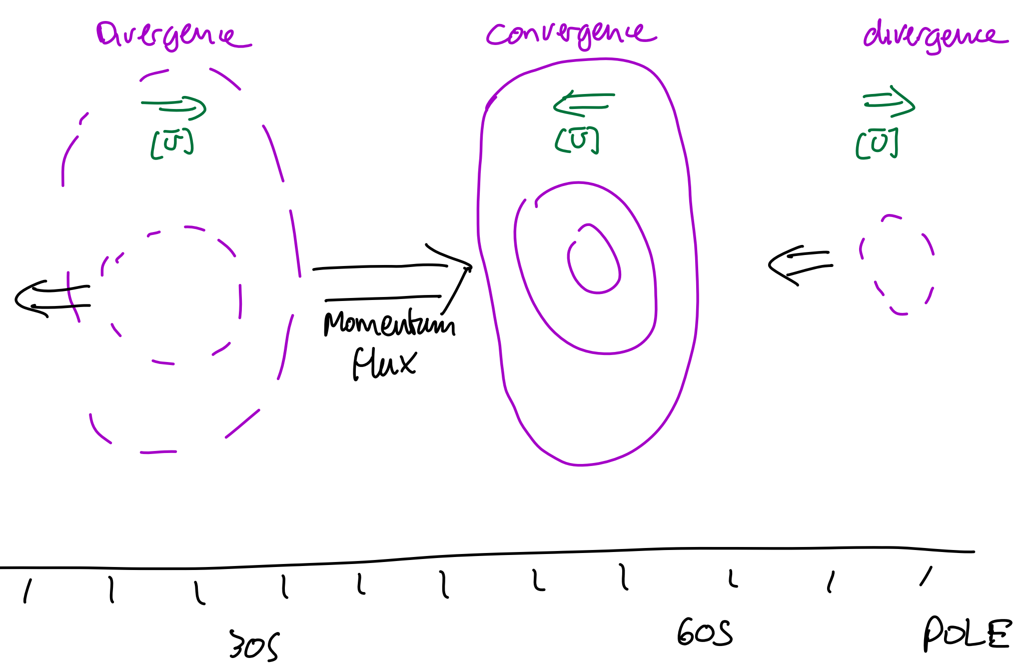
$$S = -\frac{1}{R_e \cos^2\phi}\frac{\partial\left([\overline{u'v'}]\cos^2\phi\right)}{\partial\phi} - \frac{\partial\left([\overline{u'\omega'}]\right)}{\partial p} \qquad \text{transient}$$

$$-\frac{1}{R_e \cos^2\phi}\frac{\partial\left([\bar{u}^*\bar{v}^*]\cos^2\phi\right)}{\partial\phi} - \frac{\partial\left([\bar{u}^*\bar{\omega}^*]\right)}{\partial p} \qquad \text{stationary}$$

In midlatitudes $\left|[\bar{\zeta}]\right| \ll f$

$$\Rightarrow f[\bar{v}] \approx -S$$

Eddy fluxes determine meridional flow!



Angular-momentum budget of the column

The above argument breaks down if we try to apply it to the whole column:

Integrate from surface to Top of atmosphere:

$$\int_0^{p_s} f[\bar{v}]\, dp = \int_0^{p_s} -S\, dp$$

Remember, by mass conservation

$$\int_0^{\infty} [\overline{\rho v}]\, dz = 0 \qquad \text{or} \qquad \int_0^{p_s} [\bar{v}]\, dp = 0$$

But at midlatitudes $\displaystyle\int_0^{p_s} [\bar{S}]\, dp > 0$

What is going wrong?

We are missing friction!

- at low levels, friction is important and acts to close the circulation
- eddies in the upper troposphere control this frictional flow
- This is the "downward control" principle

Friction and the angular-momentum budget

Angular momentum budget in flux form:

$$\frac{\partial}{\partial t}(M_r) + \nabla\cdot(\underline{u}M_r) = -\frac{\partial \Phi}{\partial \lambda} + R_e\cos\phi\,(fv + F_\lambda) \qquad (1)$$

Must now consider friction F_λ. Can write friction as divergence of (viscous) stress:

$$\rho \underset{\sim}{F} = -\nabla\cdot\underset{\approx}{\mathbb{P}} \qquad \text{(in height co-ords)}$$

tensor with 9 elements τ_{ij}

ith component of $\nabla\cdot\underset{\approx}{\mathbb{P}} = \partial_i P_{ij}$

Einstein notation

Zonal component of frictional force:

$$\rho F_\lambda = \frac{-1}{R_e\cos\phi}\left(\frac{\partial \mathbb{P}_{\lambda\lambda}}{\partial\lambda} + \frac{\partial(\mathbb{P}_{\lambda\phi}\cos\phi)}{\partial\phi}\right) - \frac{\partial \mathbb{P}_{\lambda z}}{\partial z}$$

Generally dominated by vertical stress:

$$\rho F_\lambda \simeq -\frac{\partial P_{\lambda z}}{\partial z} = \rho g\frac{\partial P_{\lambda z}}{\partial p}$$

Equation (1) then becomes:

$$\frac{\partial}{\partial t}(M_r) + \nabla\cdot(\underline{u}M_r) = \underset{(c)}{-\frac{\partial\Phi}{\partial\lambda}} + \underset{(a)}{R_e\cos\phi\, fv} + \underset{(b)}{g R_e\cos\phi\,\frac{\partial \mathbb{P}_{\lambda z}}{\partial p}} \qquad (2)$$

Vertically integrate (2) over the depth of the atmosphere with mass weighting and take the zonal and time mean.

Remember: $\rho dz \simeq \frac{1}{g} dp$ (Hydrostatic balance)

The LHS is then given by,

$$\text{LHS} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\tau}\int_0^{\tau} \left\{ \int_0^{p_s} \frac{\partial M_r}{\partial t} + \nabla\cdot(\underline{u} M_r)\,\frac{dp}{g} \right\} \underbrace{dt}_{\text{time average}}\ \underbrace{d\lambda}_{\text{zonal average}}$$

Note that since $p_s = p_s(\lambda, \phi, t)$, must perform vertical integral first

However, if we define $\underline{u} = 0$ for $p > p_s$, we have that

$$\int_0^{p_s} \frac{\partial M_r}{\partial t}\, dp = \int_0^{p_{00}} \frac{\partial M_r}{\partial t}\, dp$$

where p_{00} is a constant pressure such that

$$p_{00} > p_s \quad \forall\ \lambda, \phi, t$$

We then have that the left-hand side of (2) may be written,

$$\text{LHS} = \int_0^{p_{00}} \left\{ \left[\overline{\frac{\partial M_r}{\partial t}}\right] + \nabla\cdot\left[\overline{\underline{u} M_r}\right] \right\} \frac{dp}{g}$$

Assuming there are no secular trends in M_r, the first term is zero.

Thus we have,

$$\text{LHS} = \int_0^{p_{00}} \frac{1}{R_e\cos\phi} \frac{\partial}{\partial\phi}\left[\overline{v M_r}\right]\cos\phi\, \frac{dp}{g} + \underbrace{\int_0^{p_{00}} \frac{\partial\left[\overline{\omega M_r}\right]}{\partial p}\, \frac{dp}{g}}_{\substack{= \left[\omega M_r\right]_0^{p_{00}} \\ = 0}}$$

since $\omega = 0$ at $p = p_{00}$ and $p = 0$

$$\text{LHS} = \int_0^{p_{00}} \frac{1}{R_e\cos\phi}\frac{\partial}{\partial\phi}[\overline{vM_r}]\cos\phi \, \frac{dp}{g}$$

Divergence of the meridional flux of angular momentum

On the RHS we have 3 terms

(a) Coriolis term:

$$a = \int_0^{p_{00}} R_e\cos\phi f [\bar{v}] \, dp$$

This term is zero by mass conservation:

$$\int_0^{p_{00}} [\bar{v}] \frac{dp}{g} = \text{total meridional mass flux at a given latitude}$$

(b) Surface friction:

$$b = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\tau}\int_0^{\tau} \left(\int_0^{p_s} R_e\cos\phi \frac{\partial \mathbb{P}_{\lambda z}}{\partial p} \, dp \right) dt \, d\lambda$$

Now
$$\int_0^{p_s} \frac{\partial \mathbb{P}_{\lambda z}}{\partial p} \, dp = \left(\mathbb{P}_{\lambda z}(p = p_s) - \mathbb{P}_{\lambda z}(p=0) \right)$$
$$= \mathbb{P}_s$$

surface frictional stress

Thus we have,

$$b = R_e\cos\phi \, [\overline{\mathbb{P}_s}]$$

(c) Pressure gradient term:

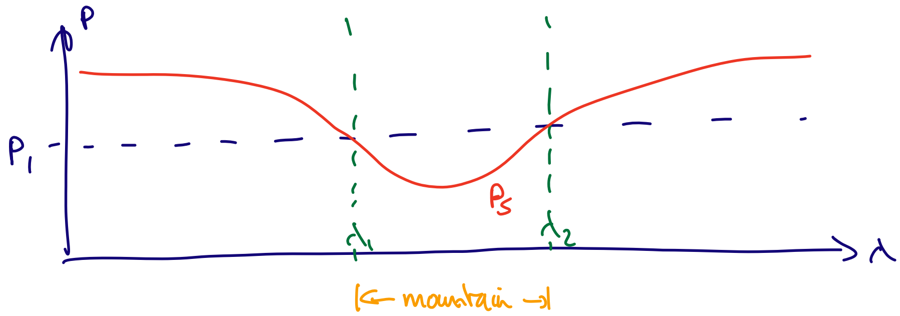
$$C = -\frac{1}{\tau}\int_0^\tau \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{p_s} \frac{\partial \Phi}{\partial \lambda}\,\frac{dp}{g}\right) d\lambda\, dt$$

Consider $\displaystyle\int_0^{2\pi}\int_0^{p_s} \frac{\partial \Phi}{\partial \lambda}\,\frac{dp}{g}\,d\lambda = \int_0^{p_{00}}\int_0^{2\pi} \frac{\partial \Phi}{\partial \lambda}\,H(p_s - p)\,d\lambda\,dp$

Heaviside function

This is zero, provided $p_1 < p_s \ \forall \lambda$

But suppose the surface outcrops at some latitude:



Then we must split the integral

$$\int_0^{2\pi} \frac{\partial \Phi}{\partial \lambda} H(p_s - p)\,d\lambda = \int_0^{\lambda_1} \frac{\partial \Phi}{\partial \lambda}\,d\lambda + \int_{\lambda_2}^{2\pi} \frac{\partial \Phi}{\partial \lambda}\,d\lambda$$

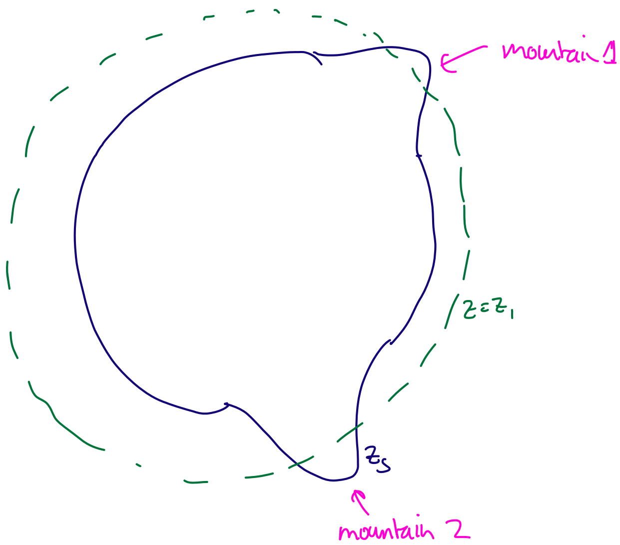
$$= \Phi(\lambda_1) - \cancel{\Phi(0)} + \cancel{\Phi(2\pi)} - \Phi(\lambda_2)$$

$$= \Phi(\lambda_1) - \Phi(\lambda_2)$$

$$= \Phi_W - \Phi_E$$

What happens if there is more than one mountain?

$\Rightarrow$ Need to sum over multiple integrals!



Thus we have,

$$\int_0^{2\pi} \frac{\partial \Phi}{\partial \lambda} d\lambda = \sum_i \Phi_W^i - \Phi_E^i \quad , \quad \text{for} \quad i = 1, \dots, n$$

This represents a form drag on the atmosphere by mountains.

We can also express the form drag term more simply.

Remember the Liebnitz integral rule:

$$\frac{d}{dx}\int_0^{b(x)} f(x,t)\,dt = f(x, b(x))\, b'(x) + \int_0^{b(x)} \frac{\partial f}{\partial x}\,dt$$

Apply this to our form drag term:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial \lambda}\left(\int_0^{p_s} \Phi \,\frac{dp}{g}\right) d\lambda = \frac{1}{2\pi}\int_0^{2\pi} \left\{ \Phi(p_s)\,\frac{\partial p_s}{\partial \lambda} + \int_0^{p_s} \frac{\partial \Phi}{\partial \lambda}\,\frac{dp}{g} \right\} d\lambda$$

Now, LHS equals zero by periodicity, so we have that

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{p_s} \frac{\partial \Phi}{\partial \lambda}\,\frac{dp}{g}\,d\lambda = -\frac{1}{2\pi}\int_0^{2\pi} \frac{\Phi(p_s)}{g}\,\frac{\partial p_s}{\partial \lambda}\,d\lambda = -\left[z_s \frac{\partial p_s}{\partial \lambda}\right]$$

By product rule $\quad \left[\frac{\partial}{\partial \lambda}(z_s p_s)\right] = \left[z_s \frac{\partial p_s}{\partial \lambda} + p_s \frac{\partial z_s}{\partial \lambda}\right] = 0$

$$-\left[z_s \frac{\partial p_s}{\partial \lambda}\right] = \left[p_s \frac{\partial z_s}{\partial \lambda}\right]$$

pressure force   slope of mountain in longitude

Form drag is proportional to the difference in pressure on the western & Eastern slopes of mountains

Hence we may write the form drag term

$$C = -\frac{1}{\tau}\int_0^{\tau} \frac{1}{2\pi}\int_0^{2\pi} \left(\int_0^{p_s} \frac{\partial \Phi}{\partial \lambda}\,\frac{dp}{g}\right) d\lambda\, dt$$

$$= -\left[\overline{p_s \frac{\partial z_s}{\partial \lambda}}\right]$$

Putting it all together:

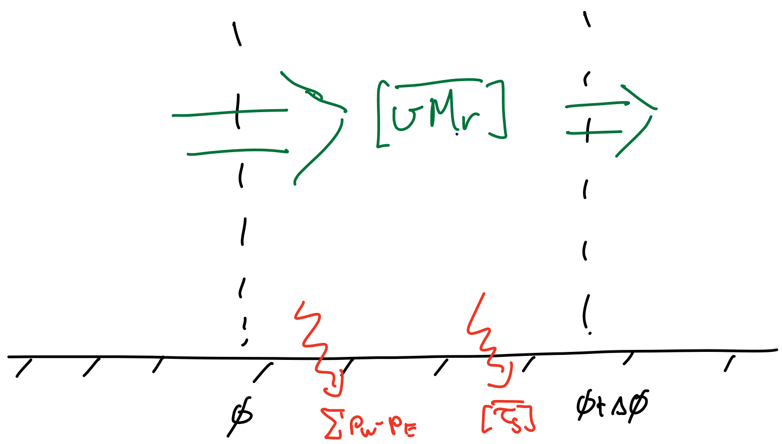
$$\int_0^{p_{00}} \frac{1}{R_e\cos\phi}\frac{\partial}{\partial\phi}\left[\overline{vM_r}\right]\cos\phi\,\frac{dp}{g} = R_e\cos\phi\left[\overline{\mathbb{P}_s}\right] - \left[\overline{p_s\frac{\partial z_s}{\partial\lambda}}\right]$$

Divergence of angular momentum

is balanced by

friction

form drag from mountains



Note that, since $M_r = R_e\cos\phi\, u$

$vM_r = R_e\cos\phi\, uv$

And, neglecting mountain drag:

$$\Rightarrow \frac{1}{\cos^2\phi}\frac{\partial}{\partial\phi}\int_0^{p_s}\left[\overline{uv}\right]\cos^2\phi\,\frac{dp}{g} = \left[\overline{\tau_s}\right]$$

- Remember: $[\overline{uv}]$ is dominated by $[\overline{u'v'}]$
- Convergence of momentum flux

$$\text{LHS} < 0$$
$$[\tau_s] < 0$$

$\Rightarrow$ Friction is a westward force

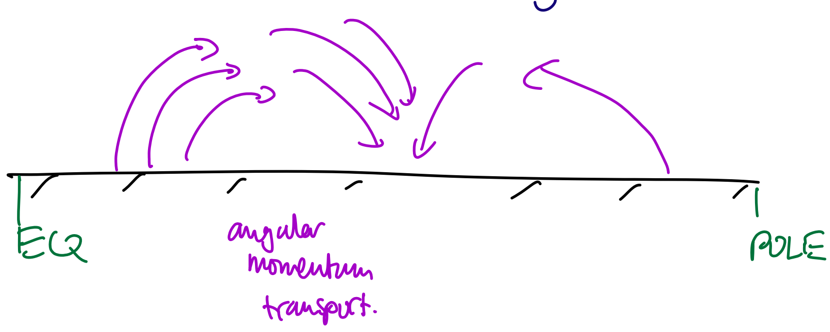
$\rightarrow$ surf. wind must be westerly!

Midlatitude westerlies are a direct result of eddy momentum flux convergence.

Atmospheric angular momentum cycle

Preceding analysis paints a picture of angular momentum transport:

- Produced by surface in region of Easterlies (tropics, polar)
- transported to upper troposphere and converged to midlatitudes
- removed at surface by friction + mountain torques



Note that angular momentum is being converged into the jet.

Angular-momentum transport is upgradient!

Global Angular momentum balance

Consider again the relative angular momentum equation

$$\frac{\partial}{\partial t}(\rho M_r) + \nabla\cdot(\rho \underset{\sim}{u} M_r) = -\frac{\partial p}{\partial \lambda} + R_e\cos\phi\left(\rho f v - \frac{\partial \tau_{\lambda z}}{\partial z}\right)$$

Integrating over the entire atmosphere, we have

$$\frac{d}{dt}\left\{\oint_{V_{atm}} \rho M_r \, dV\right\} = -\int_{\partial V_{atm}} \left(p_s \frac{\partial z_s}{\partial \lambda}\right) dS + \oint_{\partial V_{atm}} R_e\cos\phi \, \tau_s \, dS$$

angular momentum of the atmosphere

form drag

surface friction

⇒ The change in angular momentum depends on the total form drag and friction

⇒ In steady state there must be no net frictional force on the atmosphere

- Easterlies & westerlies must balance (as Hadley knew!)

⇒ When atmospheric angular momentum changes, this must be associated with opposite change in Earth's angular momentum.

- This is reflected in the length of day.

Summary

- Upper tropospheric zonal-mean meridional flow is determined by eddy (angular) momentum fluxes

$$f[\bar{v}] \simeq -\frac{\partial}{\partial y}[\overline{u'v'}]$$

(tangent-plane, QG approx)

- Produces thermally indirect Ferrel cell with frictional return flow near surface

"Downward control"

- Surface winds are determined by vertically integrated eddy (angular) momentum fluxes

$$\int_0^{p_s} \frac{\partial}{\partial y}[\overline{u'v'}]\,\frac{dp}{g} \simeq [\overline{\tau_s}]$$

westerlies in region of eddy-momentum flux convergence

⇒ But what sets the patterns of eddy momentum fluxes?