Lecture 6 : Analysis techniques

Decomposing the circulation

We are interested in the time- and zonal-mean circulation, the deviations from this mean, and how the two interact.

Define time mean

$$\bar{A} = \frac{1}{\tau}\int_0^{\tau} A \, dt$$

and deviations: $A' = A - \bar{A}$

Note that $\overline{A'} = \overline{A - \bar{A}} = \bar{A} - \bar{A} = 0$

τ is usually seasonal or longer timescale

Also define zonal mean

$$[A] = \frac{1}{2\pi}\int_0^{2\pi} A \, d\lambda$$

$$A^* = A - [A]$$

$$[A^*] = 0$$

Why do this?

Useful for analysing covariances

$$AB = (\bar{A} + A')(\bar{B} + B')$$

$$= \bar{A}\bar{B} + \bar{A}B' + \bar{B}A' + A'B'$$

Take the time mean:

$$\overline{AB} = \underbrace{\overline{\bar{A}\bar{B}}}_{=\bar{A}\bar{B}} + \underbrace{\overline{\bar{A}B'}}_{=\bar{A}\overline{B'}=0} + \cancelto{0}{\overline{\bar{B}A'}} + \overline{A'B'}$$

$$\overline{AB} = \bar{A}\bar{B} + \overline{A'B'}$$

covariance!

Example I

Consider the conservation equation for scalar γ

$$\frac{D\gamma}{Dt} = 0$$

Write this in flux form.

Use tangent plane equations in pressure coordinates for simplicity:

$$\frac{\partial \gamma}{\partial t} + \underset{\sim}{u} \cdot \nabla_p \gamma = \frac{\partial \gamma}{\partial t} + u\frac{\partial \gamma}{\partial x} + v\frac{\partial \gamma}{\partial y} + \omega\frac{\partial \gamma}{\partial p} = 0 \qquad (1)$$

Also have mass continuity:

$$\nabla_p \cdot \underset{\sim}{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \qquad (2)$$

Add (1) & $\gamma \times (2)$:

$$\frac{\partial \gamma}{\partial t} + \underset{\sim}{u} \cdot \nabla_p \gamma + \gamma \nabla_p \cdot \underset{\sim}{u} = 0$$

$$\frac{\partial \gamma}{\partial t} + \nabla_p \cdot (\underset{\sim}{u}\gamma) = 0 \qquad \Rightarrow \text{Flux form}$$

Now take zonal mean:

$$\frac{\partial[\gamma]}{\partial t} + \nabla_p \cdot [\underset{\sim}{u}\gamma] = 0$$

also $\nabla_p \cdot [\underset{\sim}{u}] = \frac{\partial[u]}{\partial x} + \frac{\partial[\omega]}{\partial p} = 0$ [λ-derivative is zero]

Now use our decomposition:

$$[\underset{\sim}{u}\gamma] = [\underset{\sim}{u}][\gamma] + [\underset{\sim}{u}^* \gamma^*]$$

$$\frac{\partial[\gamma]}{\partial t} + \nabla_p \cdot [\underset{\sim}{u}][\gamma] = -\nabla_p \cdot [\underset{\sim}{u}^*\gamma^*]$$

by zonal-mean continuity

$$\Rightarrow \frac{\partial[\gamma]}{\partial t} + [\underset{\sim}{u}] \cdot \nabla_p[\gamma] + [\gamma]\cancel{\nabla_p \cdot [\underset{\sim}{u}]} = -\nabla_p \cdot [\underset{\sim}{u}^*\gamma^*]$$

$$\boxed{\frac{\partial}{\partial t}[\gamma] + [\underset{\sim}{u}] \cdot \nabla_p[\gamma] = -\frac{\partial}{\partial y}[u^*\gamma^*] - \frac{\partial}{\partial p}[\omega^*\gamma^*]}$$

λ-derivative of divergence is zero

Can do this derivation in spherical coordinates

Remember $\frac{D\gamma}{Dt} = \frac{\partial \gamma}{\partial t} + \underset{\sim}{u} \cdot \nabla_p \gamma = \frac{\partial \gamma}{\partial t} + \frac{u}{R_e\cos\phi}\frac{\partial \gamma}{\partial \lambda} + \frac{v}{R_e}\frac{\partial \gamma}{\partial \phi} + \omega\frac{\partial \gamma}{\partial p}$

$$\Rightarrow \frac{\partial \gamma}{\partial t} + \frac{u}{R_e\cos\phi}\frac{\partial \gamma}{\partial \lambda} + \frac{v}{R_e}\frac{\partial \gamma}{\partial \phi} + \omega\frac{\partial \gamma}{\partial p} = 0 \qquad (1)$$

Also, from continuity: $\frac{1}{R_e\cos\phi}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial v\cos\phi}{\partial \phi}\right] + \frac{\partial \omega}{\partial p} = 0 \qquad (2)$

Take (1) + $\gamma \times$ (2):

$$\frac{\partial \gamma}{\partial t} + \frac{u}{R_e\cos\phi}\frac{\partial \gamma}{\partial \lambda} + \frac{\gamma}{R_e\cos\phi}\frac{\partial u}{\partial \lambda} + \frac{v}{R_e}\frac{\partial \gamma}{\partial \phi} + \frac{\gamma}{R_e\cos\phi}\frac{\partial v\cos\phi}{\partial \phi} + \gamma\frac{\partial \omega}{\partial p} + \omega\frac{\partial \gamma}{\partial p} = 0$$

$$\frac{\partial \gamma}{\partial t} + \frac{1}{R_e\cos\phi}\left[\frac{\partial}{\partial \lambda}(u\gamma)\right] + \frac{1}{R_e\cos\phi}\left[\frac{\partial}{\partial \phi}\gamma v\cos\phi\right] + \frac{\partial}{\partial p}(\gamma\omega) = 0$$

$$\frac{\partial \gamma}{\partial t} + \frac{1}{R_e\cos\phi}\left[\frac{\partial(u\gamma)}{\partial\lambda} + \frac{\partial v\gamma\cos\phi}{\partial\phi}\right] + \frac{\partial(\omega\gamma)}{\partial p} = 0$$

$$\frac{\partial \gamma}{\partial t} + \nabla_p\cdot(\underset{\sim}{u}\gamma) = 0$$

Take zonal mean

$$\frac{\partial}{\partial t}[\gamma] + \frac{1}{R_e\cos\phi}\frac{\partial}{\partial\phi}[v\gamma]\cos\phi + \frac{\partial}{\partial p}[\omega\gamma]$$

The λ-derivative of the divergence is zero.

Remember $[v\gamma] = [v][\gamma] + [v^*\gamma^*]$

$$\Rightarrow \frac{\partial[\gamma]}{\partial t} + \frac{1}{R_e\cos\phi}\frac{\partial}{\partial\phi}[v][\gamma]\cos\phi + \frac{\partial}{\partial p}[\omega][\gamma] = \frac{-1}{R_e\cos\phi}\frac{\partial}{\partial\phi}[v^*\gamma^*]\cos\phi - \frac{\partial}{\partial p}[\omega^*\gamma^*]$$

Expand LHS:

$$\frac{\partial[\gamma]}{\partial t} + \frac{[v]}{R_e}\frac{\partial[\gamma]}{\partial\phi} + [\omega]\frac{\partial[\gamma]}{\partial p} = \frac{-1}{R_e\cos\phi}\frac{\partial}{\partial\phi}[v^*\gamma^*]\cos\phi - \frac{\partial}{\partial p}[\omega^*\gamma^*]$$

$$+ \frac{[\gamma]}{R_e\cos\phi}\frac{\partial[v]\cos\phi}{\partial\phi} + [\gamma]\frac{\partial[\omega]}{\partial p}$$

Now, the zonal-mean continuity equation may be written,

$$\nabla_p\cdot[\underset{\sim}{u}] = 0$$

$$\Rightarrow \frac{1}{R_e\cos\phi}\frac{\partial[v]\cos\phi}{\partial\phi} + \frac{\partial[\omega]}{\partial p} = 0$$

Hence the equation for the zonal-mean γ may be written:

$$\frac{\partial[\gamma]}{\partial t} + \frac{[v]}{R_e}\frac{\partial[\gamma]}{\partial\phi} + [\omega]\frac{\partial[\gamma]}{\partial p} = \frac{-1}{R_e\cos\phi}\frac{\partial[v^*\gamma^*]}{\partial\phi} - \frac{\partial}{\partial p}[\omega^*\gamma^*]$$

$$\frac{\partial[\gamma]}{\partial t} + [\underset{\sim}{u}]\cdot\nabla_p[\gamma] = -\nabla\cdot[\underset{\sim}{u}^*\gamma^*]$$

Equation for the zonal mean:

$$\frac{\partial \gamma}{\partial t} + \underset{\sim}{u} \cdot \nabla_p \gamma = 0$$

time rate of change of γ

advection

$$\frac{\partial [\gamma]}{\partial t} + [\underset{\sim}{u}] \cdot \nabla_p [\gamma] = - \nabla_p \cdot [\underset{\sim}{u}^* \gamma^*]$$

time rate of change of $[\gamma]$

mean advection

eddy flux divergence

This is super important to understand!

Example II

time	q (kg/kg)	v (ms^{-1})	ρ ($kg\,m^{-3}$)
1	0.01	3	1
2	0.002	-1	1
3	0.014	6	1

1. Calculate meridional flux of humidity

$$\text{Flux} = \rho_0 \overline{vq}$$

{units: $ms^{-1} \cdot kg/kg \cdot kg\,m^{-3}$
$= kg\,m^{-2}s^{-1}$

more accurately a flux density
(units in p-coords are more confusing)

$$= \left[0.01(3) + 0.002(-1) + 0.014(6)\right]\frac{1}{3}$$
$$= 0.037 \quad kg\,m^{-2}s^{-1}$$

Flux by mean circulation

$$F_{MEAN} = \rho_0 \bar{v}\bar{q} = \rho_0 \left(\frac{3+6-1}{3}\right)\left(\frac{0.01+0.002+0.014}{3}\right)$$
$$= 0.023 \quad kg\,m^{-2}s^{-1}$$

Flux by eddies

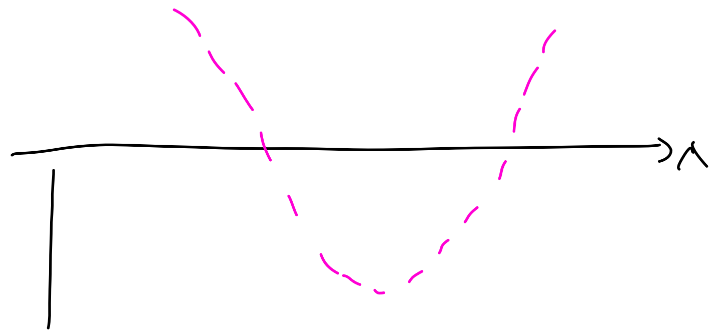
$$F_{EDDY} = \rho_0 \overline{v'q'} = \rho_0 \overline{vq} - \rho_0 \bar{v}\bar{q} = 0.014 \quad kg\,m^{-2}s^{-1}$$

Example III

$$[F] = \rho_0 [vq]$$

$v = \Delta V \cos\lambda$
$q = q_0 + \Delta q \cos\lambda$

v, q



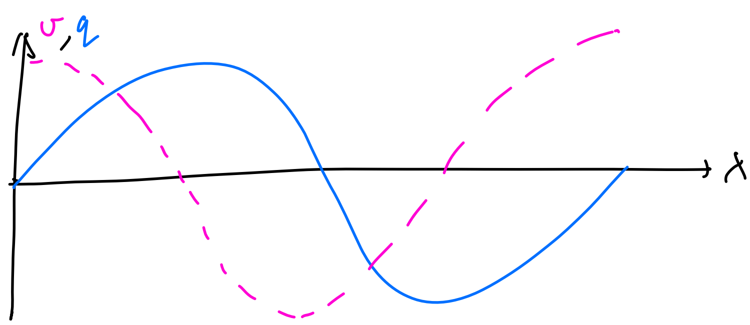
Note that $\bar{v} = 0$, $\bar{q} = 0$

$$[F] = \frac{\rho_0 \Delta q \Delta v}{2\pi} \int_0^{2\pi} \cos(\lambda)\cos(\lambda)\, d\lambda = \frac{\rho_0 \Delta q \Delta v}{2\pi} \int_0^{2\pi} \cos^2(\lambda)\, d\lambda > 0$$

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^2(\lambda)\, d\lambda = \frac{1}{2\pi} \int_0^{2\pi} 1 + \frac{\cos(2\lambda)}{2}\, d\lambda = \frac{1}{2}$$

$$\Rightarrow [F] = \frac{\rho_0 \Delta q \Delta v}{2}$$

Now suppose $q = \sin \lambda$



$$[F] = \frac{\rho_0 \Delta q \Delta v}{2} \int_0^{2\pi} \cos(\lambda) \sin(\lambda)\, d\lambda$$

Let $u = \sin(\lambda)$

$$[F] = \frac{1}{2\pi} \int u\, du = \frac{1}{2\pi} \left[\frac{\sin^2 \lambda}{2} \right]_0^{2\pi} = 0$$

Combining the space & time operator

Consider the temporal covariance: $\overline{AB}$

We know that

$$\overline{AB} = \bar{A}\bar{B} + \overline{A'B'}$$

Now, take the zonal mean

$$[\overline{AB}] = [\bar{A}\bar{B}] + [\overline{A'B'}]$$

Decompose the first term into zonal mean and eddy:

$$[\overline{AB}] = \underbrace{[\bar{A}][\bar{B}]}_{\text{zonal- and time-mean}} + \underbrace{[\bar{A}^*\bar{B}^*]}_{\text{stationary eddies}} + \underbrace{[\overline{A'B'}]}_{\text{transient eddies}}$$

Could also go further:

$$[\overline{AB}] = [\bar{A}][\bar{B}] + [\bar{A}^*\bar{B}^*] + \overline{[A'][B']} + [\overline{A'^*B'^*}]$$

But usually do not separate transient asymmetric from transient symmetric.

Example

Take $A = u$, $B = u$

$$[\overline{u^2}] = [\bar{u}]^2 + [\bar{u}^{*2}] + [\overline{u'^2}]$$

Add to $[\overline{v^2}]$:

$$[\overline{u^2+v^2}] = [\bar{u}]^2 + [\bar{v}]^2 + [\overline{u}^{*2}] + [\overline{v}^{*2}] + [\overline{u'^2}] + [\overline{v'^2}]$$

mean kinetic energy

KE of stationary eddies

KE of transient eddies

State estimation techniques

How do we calculate $[\overline{u'^2}]$ or $[\overline{u'v'}]$ for the atmosphere?

Observations are scattered in space and time

Need a method to construct gridded estimate

Method 1 : Peixoto & Oort - objective analysis

- Used to produce figures in Peixoto & Oort (1992).
- uses radiosonde data.

Step 1

- Quality control data (remove outliers)
- Calculate means + covariances at each station

$\overline{T}$, $\overline{T'^2}$, $\overline{vT}$, $\overline{v'T'}$

Step 2

- Define an "initial guess" on a grid

- Peixoto & Oort used a zonal mean climatology in 10° latitude bands

step 3

- Update the initial guess using the available data.
- In data sparce regions solve the following BVP:

$$\nabla^2 T = F(\lambda, \phi)$$

$$F = \nabla^2 T_{guess}$$

Why not just a simple interpolation routine?

Method 2: Modern reanalysis

- Use atmospheric general circulation model to produce initial guess
- Correct initial guess using data assimilation techniques
- Basic example is the Kalman Filter
  - initial guess provided by forecast model
  - updated based on observations in order to minimise a cost function that is constructed based on estimated error in obs and model.

Next lecture: Begin to apply these techniques to understand the atmospheres angular momentum budget.