Lecture 5 : Axisymmetric Hadley Cells

Last lecture

- Showed that RCE solution is not the correct solution in the limit of weak viscosity
- In particular RCE solution implies strong meridional temperature gradients which, by thermal wind imply zonal wind distribution that violates Hide's theorem
- Large-scale circulation must therefore develop that reduces meridional temperature gradients.

- This lecture we seek to understand what this "nearly inviscid" axisymmetric circulation looks like

What does this nearly inviscid circulation look like?

Retaining our axisymmetric assumption, the equation for angular momentum for steady inviscid flow is,

$$\underset{\sim}{u} \cdot \nabla M = 0$$

$\Rightarrow$ velocity vectors must be perpendicular to ∇M

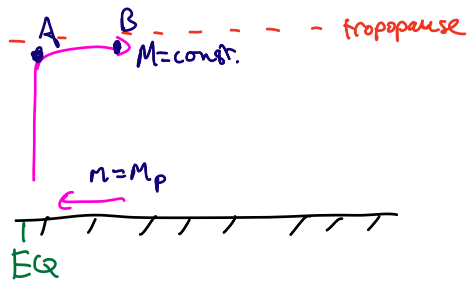
i.e., velocity is parallel to M contours

M is conserved along streamlines!

$$M = R_e \cos\phi \left[u + \Omega R_e \cos\phi \right]$$

Held-Hou model

- Originally described in Held & Hou (1980)
- They used Boussinesq equations.
- We will assume atmosphere in p-co-ords



- Assume air rises at equator, flows poleward conserving angular momentum.
- Assume surface winds are weak: What is the AM at A

$$M_t(\phi) = M_p(0)$$

tropopause angular momentum

planetary angular momentum

Hence we have

$$M_t(\phi) = \Omega R_e^2$$

$$R_e \cos\phi \,(u_t + \Omega R_e \cos\phi) = \Omega R_e^2$$

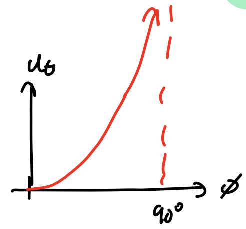
$$u_t = \frac{\Omega R_e}{\cos\phi} - \Omega R_e \cos\phi$$

$$= \frac{\Omega R_e}{\cos\phi}\left(1 - \cos^2\phi\right)$$

$$= \frac{\Omega R_e \sin^2\phi}{\cos\phi}$$

$$u_t = \Omega R_e \sin\phi \tan\phi$$

why does $u_t \to \infty$ at pole?



Now, assume we have hydrostatic & gradient wind balance. Allows us to derive thermal wind equation as in prev. lecture:

$$\left[2\Omega \sin\phi\, u + \frac{u^2 \tan\phi}{R_e}\right]_{p_s}^{p_t} = -\frac{R_d}{R_e} \ln\left(\frac{p_s}{p_t}\right) \frac{\partial \hat{T}}{\partial \phi}, \quad \hat{T} = \frac{1}{\ln\left(\frac{p_s}{p_t}\right)} \int_{\ln p_t}^{\ln p_s} T \, d\ln(p)$$

Again assuming winds near the surface are weak:

$$2\Omega R_e \sin\phi\, u_t + u_t^2 \tan\phi = -\frac{R_d}{R_e} \frac{\partial \hat{T}}{\partial \phi}$$

Substituting the angular-momentum conserving wind:

$$2\Omega^2 R_e^2 \sin^2\phi \tan\phi + \Omega^2 R_e^2 \sin^2\phi \tan^3\phi = -\frac{R_d}{R_e} \frac{\partial \hat{T}}{\partial \phi} \ln\left(\frac{p_s}{p_t}\right)$$

$$\Rightarrow -\frac{R_d}{\Omega^2 R_e^2} \ln\left(\frac{p_s}{p_t}\right) \frac{\partial \hat{T}}{\partial \phi} = \sin^2\phi \tan\phi \left(2 + \tan^2\phi\right)$$

$$= \sin^2\phi \tan\phi \left(\frac{2\cos^2\phi + \sin^2\phi}{\cos^2\phi}\right)$$

$$= \tan^3\phi \left(1 + \cos^2\phi\right)$$

Integrate,

$$\hat{T} - T_0 = -\frac{\Omega^2 R_e^2}{R_d \ln\left(\frac{p_s}{p_t}\right)} \int_0^{\phi} \tan^3\phi' \left(1 + \cos^2\phi'\right) d\phi'$$

[Google this]

Now,

$$\int_0^{\phi} \tan^3 x \,(1+\cos^2(x))\, dx = \int_0^{\phi} \frac{\tan(x)}{\cos^2 x}\,(1-\cos^2 x)(1+\cos^2 x)\, dx$$

$$= \int_0^{\phi} \frac{\tan(x)}{\cos^2 x} \left\{ 1 - \cos^4(x) \right\} dx$$

$$= \int_0^{\phi} \frac{\tan(x)}{\cos^2(x)}\, dx \;+\; \int_0^{\phi} \sin(x)\cos(x)\, dx$$

$$= \left[\frac{\tan^2(x)}{2} \right]_0^{\phi} - \left[\frac{\sin^2(x)}{2} \right]_0^{\phi}$$

$$= \frac{1}{2} \left\{ \frac{\sin^2\phi - \sin^2\phi\cos^2\phi}{\cos^2\phi} \right\}$$

$$= \frac{1}{2} \left\{ \frac{\sin^2\phi - \sin^2\phi(1-\sin^2\phi)}{\cos^2\phi} \right\}$$

$$= \frac{\sin^4\phi}{2\cos^2\phi}$$

Hence, $$\hat{T}(\phi) = T_0 - \frac{\Omega^2 R_e^2}{2 R_d \ln\left(\frac{p_s}{p_t}\right)} \left(\frac{\sin^4\phi}{\cos^2\phi} \right)$$

Have found mean temperature distribution using only angular momentum conservation!

Held-Hou solution characterised by:

- flat temperature dist. near equator ($\hat{T} \sim \phi^4$)
- As approach pole $u_t \to \infty$ unphysical! Why?
- At some latitude must transition to RCE.

Thermodynamics of Hadley cell

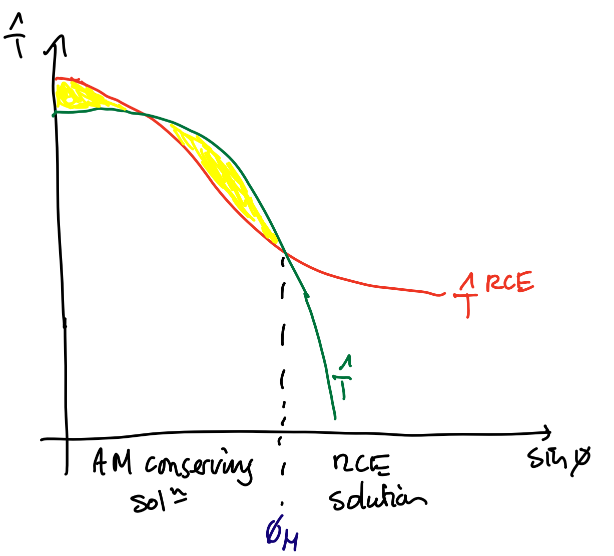
- Expect Hadley cell to transport energy poleward
- If no eddies, Hadley cell heat flux divergence must balance heat fluxes due to radiation/convection.
- Consider simple parameterisation:

$$\langle Q \rangle = -\frac{\hat{T} - \hat{T}^{RCE}}{\tau}$$

Linear relaxation to RCE state.

Implies: $$\int_{globe} \hat{T} \, dS = \int_{globe} \hat{T}^{RCE} \, dS$$

This gives us an equal area constraint



Calculating the Hadley cell extent

We have 2 constraints:

1. Continuity of mean temp. accross the Hadley cell edge:

Define ϕ_H as latitude of Hadley cell edge:

$$\hat{T}(\phi_H) = \hat{T}^{RCE}(\phi_H)$$

2. Energy conservation:

$$\int_0^{\phi_H} (\hat{T} - \hat{T}^{RCE}) \cos\phi \, d\phi = 0$$

Can solve for ϕ_H and S_0

For simplicity, use small angle approx.

$\Rightarrow \quad \sin\phi \simeq \phi \quad , \quad \cos\phi \simeq 1$

$$\hat{T} = \hat{T}_0 - \frac{\Omega^2 R_e^2}{2R_d} \phi^4 = \hat{T}_0 - \delta\hat{T} \phi^4$$

$$\delta\hat{T} = \frac{\Omega^2 R_e^2}{2R_d}$$

$$\hat{T}^{RCE} = \hat{T}_0^{RCE} - \delta\hat{T}^{RCE} \phi^2$$

Constraint (2) gives:

$$\int_0^{\phi_H} \hat{T}_0 - \hat{T}_0^{RCE} - \delta\hat{T}\phi^4 + \delta\hat{T}^{RCE}\phi^2 \, d\phi = 0$$

$$\Rightarrow \quad (\hat{T}_0 - \hat{T}_0^{RCE})\phi_H - \frac{\delta\hat{T}\phi_H^5}{5} + \frac{\delta\hat{T}^{RCE}\phi^3}{3} = 0$$

$$\frac{\delta\hat{T}}{5}\phi_H^4 - \frac{\delta\hat{T}^{RCE}}{3}\phi_H^2 - (\hat{T}_0 - \hat{T}_0^{RCE}) = 0 \qquad (1)$$

Constraint (1) gives

$$\hat{T}_0 - \delta\hat{T}\phi_H^4 - \hat{T}_0^{RCE} + \delta\hat{T}^{RCE}\phi_H^2 = 0 \qquad (2)$$

add (1) and (2)

$$\left(\frac{\delta\hat{T}}{5} - \delta\hat{T}\right)\phi_H^4 + \left(\delta\hat{T}^{RCE} - \frac{\delta\hat{T}^{RCE}}{3}\right)\phi_H^2 = 0$$

$$\frac{4\delta\hat{T}}{5}\phi_H^2 = \frac{2\delta\hat{T}^{RCE}}{3}$$

$$|\phi_H| = \left(\frac{5}{6}\frac{\delta\hat{T}^{RCE}}{\delta\hat{T}}\right)^{\frac{1}{2}}$$

$$|\phi_H| = \left(\frac{5}{3}\delta\hat{T}^{RCE}\frac{R_d\ln\left(\frac{p_s}{p_t}\right)}{\Omega^2 R_e^2}\right)^{\frac{1}{2}}$$

Let's plug in some reasonable numbers:

$p_s/p_t \approx 10 \Rightarrow \ln\left(\frac{p_s}{p_t}\right) = 2.3$

$\delta\hat{T}^{RCE} \approx (100\text{ K})$

$$|\phi_H| = \frac{5}{3} \times \frac{100 \times (2.3) \times 287}{(7.29\times10^{-5})^2 \times (6.3\times10^6)^2}$$

$$= 0.5$$

$$\simeq 30^\circ$$

Surprisingly close to observed!

Strength of the Hadley cell

In the Held-Hou theory, the strength of the Hadley cell is related to its energy transport.

In particular, we assumed that

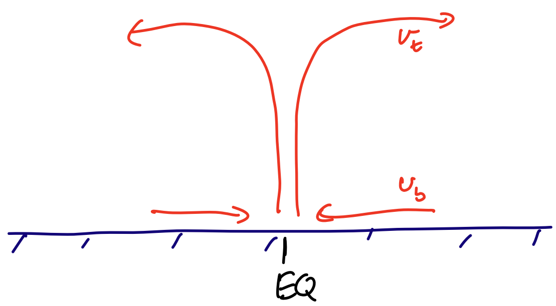
1) The Hadley cell is energetically closed
2) The net column heating is proportional the deviation of the temperature from its RCE value.

Together, these assumptions imply

$$\frac{1}{R_e \cos\phi} \frac{\partial F_H \cos\phi}{\partial \phi} = \langle Q \rangle = \frac{\hat{T} - \hat{T}^{RCE}}{\tau}$$

where F_H is the energy flux by the Hadley cell

Held & Hou assumed poleward flow near the tropopause and equatorward return flow



Assuming the flow is concentrated in a layers near the tropopause and surface of pressure depth δp:

$$F_H = \left(\int_{p_t - \delta p}^{p_t} v_t \frac{dp}{g} \right) h_t - \left(\int_{p_s - \delta p}^{p_s} v_b \frac{dp}{g} \right) h_b$$

where h is the "energy" (see below).

By mass conservation, have that

$$\int_{p_t - dp}^{p_t} v_t \frac{dp}{g} = - \int_{p_s - \delta p}^{p} v_b \frac{dp}{g} = \Psi_{max}$$

$$\Rightarrow \quad F_M = \Psi_{max} \, \Delta h$$

Hadley cell strength

gross energy stratification

Gross energy stratification

- Held & Hou neglect moisture: Relevant energy variable is the dry static energy

$$h_d = c_p T + gz$$

- $\frac{\partial h_d}{\partial z}$ must be positive for gravitational stability

- Held & Hou took Δh_d as a parameter

- In moist atmosphere, relevant variable is moist static energy:

$$h_m = c_p T + gz + L_v q$$

- In quasi-equilibrium $\frac{\partial h^*}{\partial z} \simeq 0$ in troposphere

(h is related to s, entropy)

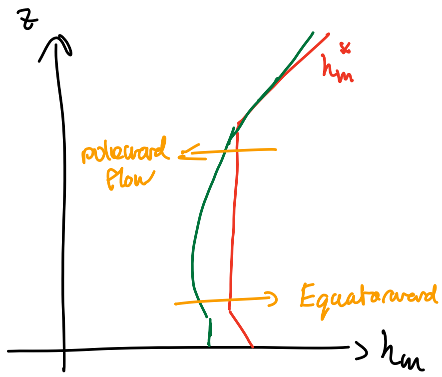
$$h_m^* = c_p T + gz + L_v q^*$$

$$h_m^* - h_m = L_v (q^* - q)$$

Boundary layer: RH high, $q^* - q$ is small

troposphere : RH lower, $q^* - q$ is large

tropopause : T is low, q^* is small $q^* - q$ is small



h_m has minimum in mid troposphere!

The size (and sign!) of the gross energy stratification depends on the profile of v!

This is usually discussed in terms of the gross moist stability introduced by Neelin & Held (1987)

Additional complication:

-> Most poleward energy transport at low latitudes is effected through ocean circulation

-> But ocean circ. driven by surf. wind stress.

Hadley cell energetics is complicated. Will return to this topic later in the course.

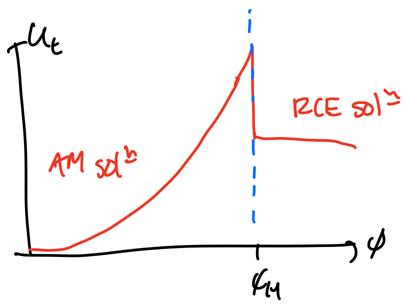
Held & Hou zonal wind distribution

- Angular momentum conserving solution in tropics
- RCE solution further poleward. (no overturning circulation)

Tropopause winds

$$u_t = \begin{cases} \Omega R_e \sin\phi \tan\phi \simeq \Omega R_e \phi^2 & |\phi| < \phi_H \\ \dfrac{R_d \ln\left(\frac{p_s}{p_t}\right)}{\Omega R_e \sin\phi} \dfrac{dT^{RCE}}{d\phi} & |\phi| > \phi_H \end{cases}$$

Results in sharp subtropical jet:



Held & Hou predicts:

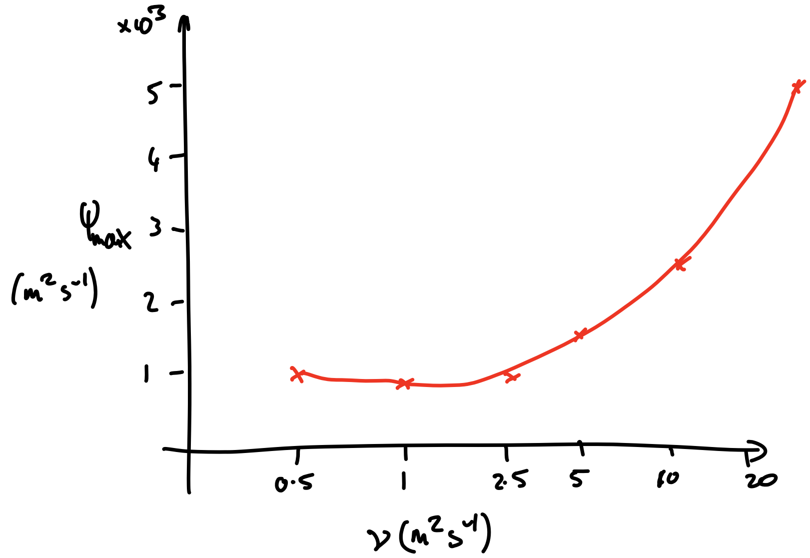
- Circulation to ~30°
- Sharp subtropical jet
- Weak overturning further poleward

These are all consistent with the observed equinoctial circulation

Problems with Held Hou

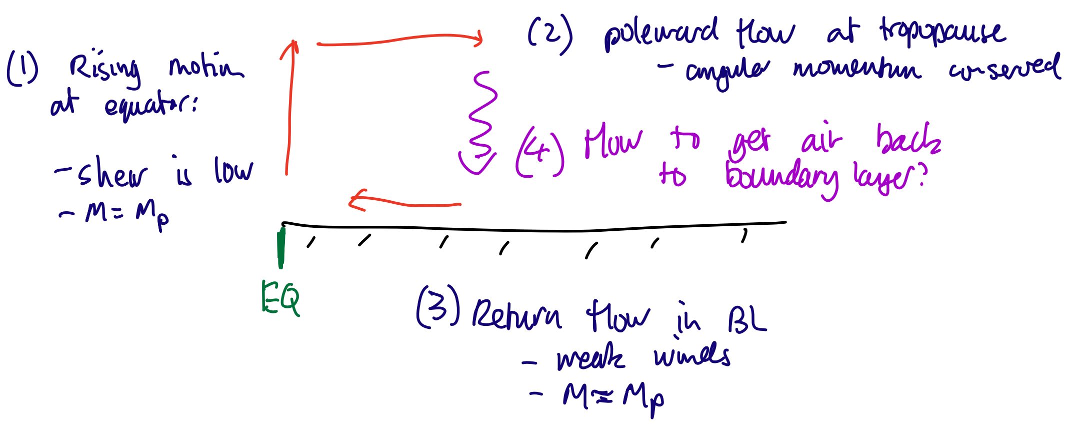
In numerical sol^{n}s of the Held-Hou model the strength of the cell depends on the viscosity.

Remarkably, the strength increases with increasing viscosity!



What is going on?

Let's revisit the assumptions of the model:

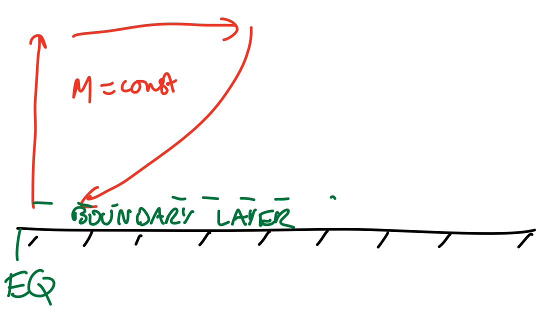


We assume air enters BL with weak winds

→ How are the winds reduced in the descending branch?

2 possibilities

1) Sloping descending branch:



Air only enters boundary layer near equator where

$M \simeq M_p$

This is not what happens in Held-Hou, nor in observations.

2) Need to remove angular-momentum in the descending branch

How?

EDDIES!