Lecture 4 : Hide's Theorem

Last time:

Described a fully non-linear solution to the governing equations under no viscosity.

Why can't such a flow exist?

To answer this, we need to think about angular momentum.

What is angular momentum

Consider zonal momentum equation:

$$\frac{Du}{Dt} = 2\Omega \sin\phi\, v + \frac{uv\tan\phi}{R_e} - \frac{1}{R_e\cos\phi}\frac{\partial \Phi}{\partial \lambda} + F_x$$

Can show (you will) that:

for axisymmetric, inviscid flow

$$M = R_e\cos\phi\{u + \Omega R_e\cos\phi\}$$

is conserved.

More specifically, for axisymmetric flow

$$\frac{DM}{Dt} = R_e\cos\phi\, F_\lambda$$

Flux form

For any conservation equation of the form

$$\frac{D\gamma}{Dt} = \frac{\partial \gamma}{\partial t} + \underline{u}\cdot\nabla\gamma = S$$

Can use continuity to write it in flux form

Multiply by density: $\rho \frac{\partial \gamma}{\partial t} + \rho \underline{u} \cdot \nabla \gamma = \rho S$

multiply continuity by γ: $\gamma \frac{\partial \rho}{\partial t} + \gamma \nabla \cdot \rho \underline{u} = 0$

Add: $\underbrace{\frac{\partial \rho \gamma}{\partial t} + \nabla \cdot (\rho \underline{u} \gamma) = \rho S}_{\text{Flux form}}$

Can write angular momentum equation in flux form:

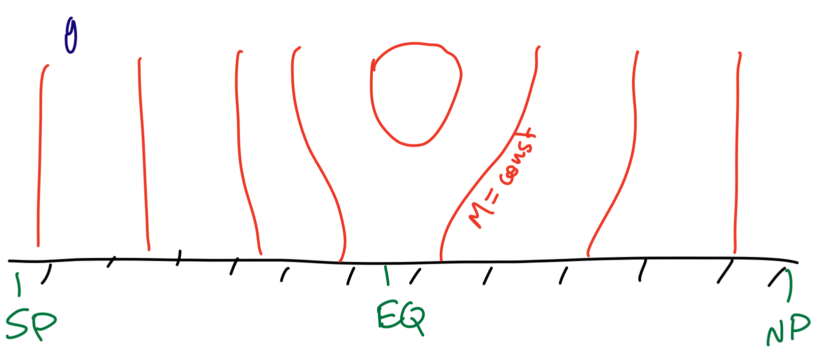
$$\frac{\partial}{\partial t}(\rho M) + \nabla \cdot (\rho \underline{u} M) = \rho R_e \cos\phi \, F_x$$

Suppose flow is steady, and friction acts as a downgradient diffusion of angular momentum

$$\nabla \cdot (\rho \underline{u} M) = \nabla (\rho \nu \nabla M) \qquad (1)$$

"viscosity"

Now, consider a flow in which there is a maximum of angular momentum



Must be able to find a closed contour of M, say $M = M_0$.
Integrate (1) over this region:

$$\oint_{M > M_0} \nabla \cdot (\rho \underline{u} M)\, dV = \oint_{M > M_0} \nabla \cdot (\rho \nu \nabla M)\, dV$$

By divergence theorem,

$$\oint_{M > M_0} \nabla \cdot (\rho \underline{u} M) = \oint_{M = M_0} (\rho \underline{u} M)\, d\underline{S}$$

But M is constant

$$\Rightarrow \oint_{M = M_0} (\rho \underline{u} M) \cdot dS = M \overbrace{\oint \rho \underline{u} \cdot dS}^{\text{What is this}} = 0$$

by steady mass conservation

Hence, require that

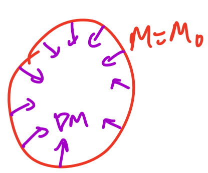
$$\oint_{M > M_0} \nabla \cdot (\rho \nu \nabla M)\, dV = 0$$

But we can apply divergence theorem here too:

$$\oint_{M > M_0} \nabla \cdot (\rho \nu \nabla M)\, dV = \oint_{M = M_0} \rho \nu \nabla M \cdot dS$$

What is $\nabla M \cdot d\underline{S}$ → where does M point?

Now, ∇M points to the maximum in M, which is contained within the contour.



Thus we have

$$\nabla M \cdot dS < 0 \quad \text{on} \quad M = M_0$$

$$\Rightarrow \oint_{M = M_0} \rho \nu \nabla M \cdot dS < 0$$

We cannot reach a steady state!

Downgradient diffusion will always remove a maximum in M, and this cannot be maintained by advection.

Solution is unattainable for any $\nu > 0$ no matter how small!

Hide's Theorem: In steady axisymmetric flow, there can be no extrema of angular momentum except at solid boundaries.

How do we tell if our solution violates Hide's Theorem?

skip

Thermal wind in a convecting atmosphere

Consider the RCE solution.

Expect atmosphere to be locally neutral to moist convection everywhere. In particular,

$$s^*(\phi, p) = s_b(\phi)$$

free tropospheric saturation entropy

boundary layer entropy

Good assumption in convecting regions of tropics.

=> In RCE atmosphere, everywhere is convecting so is reasonable assumption

RCE solution must satisfy meridional momentum equation:

In pressure co-ordinates:

$$\cancel{\frac{Dv}{Dt}}^{[1]} = -2\Omega\sin\phi\, u - \frac{u^2}{R_e}\tan\phi - \frac{1}{R_e}\frac{\partial \Phi}{\partial \phi} + \cancel{F_\lambda}^{[2]}$$

RCE solution: $v=0$ [1]

Inviscid above surface [2]

Differentiate vertically:

$$\frac{\partial}{\partial p}\left\{ 2\Omega\sin\phi\, u + \frac{u^2\tan\phi}{R_e} \right\} = \frac{-1}{R_e}\frac{\partial}{\partial\phi}\frac{\partial\Phi}{\partial p}$$

By hydrostatic balance $\quad \frac{\partial\Phi}{\partial p} = -\alpha$

$$\Rightarrow \quad \frac{\partial}{\partial p}\left\{ 2\Omega\sin\phi + \frac{u^2}{R_e}\tan\phi \right\} = \frac{1}{R_e}\frac{\partial\alpha}{\partial\phi}$$

$$\alpha = \frac{R_d T}{p}$$

$$\frac{\partial}{\partial p}\left\{ 2\Omega\sin\phi\, u + \frac{u^2}{R_e}\tan\phi \right\} = \frac{R_d}{pR_e}\frac{\partial T}{\partial\phi}$$

Integrate vertically and assuming winds near the sufrace are weak:

$$2\Omega\sin\phi\, u_t + u_t^2\frac{\tan\phi}{R_e} = \frac{R_d}{R_e}\int_{p_s}^{p_t}\frac{T}{p}\,dp$$

$$= -\frac{R_d}{R_e}\int_{\ln(p_t)}^{\ln(p_s)} T \, d\ln(p)$$

$$2\Omega\sin\phi\, u_t + u_t^2\,\frac{\tan\phi}{R_e} = -\frac{R_d}{R_e}\ln\left(\frac{p_s}{p_t}\right)\frac{\partial}{\partial\phi}\hat{T}$$

$$\hat{T} = \frac{1}{\ln\left(\frac{p_s}{p_t}\right)}\int_{\ln p_t}^{\ln p_s} T\, d\ln(p)$$

Gives a quadratic for the tropopause zonal wind!

$$u_t^2 + \left(2R_e\cos\phi\right)u_t + \frac{R_d}{\tan\phi}\ln\left(\frac{p_s}{p_t}\right)\frac{\partial\hat{T}}{\partial\phi} = 0$$

$$u_t = \frac{-2\Omega R_e\cos\phi \pm \left(4\Omega^2R_e^2\cos^2\phi - 4\frac{R_d}{\tan\phi}\ln\left(\frac{p_s}{p_t}\right)\frac{\partial\hat{T}}{\partial\phi}\right)^{\frac{1}{2}}}{2}$$

Need $u_t \to 0$ as $\frac{\partial\hat{T}}{\partial\phi} \to 0$, take +ve root:

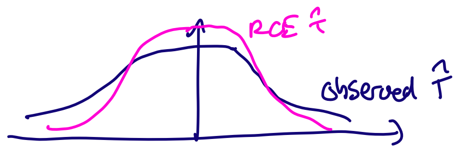
$$\frac{u_t}{\Omega R_e} = \left(\cos^2\phi - \frac{R_d}{\Omega^2R_e^2\tan\phi}\ln\left(\frac{p_s}{p_t}\right)\frac{\partial\hat{T}}{\partial\phi}\right)^{\frac{1}{2}} - \cos\phi$$

$$\frac{u_t}{\Omega R_e} = \left\{\left(1 - \frac{R_d\ln\left(\frac{p_s}{p_t}\right)}{\Omega^2R_e^2\sin\phi\cos\phi}\frac{\partial\hat{T}}{\partial\phi}\right)^{\frac{1}{2}} - 1\right\}\cos\phi$$

Relates tropopause winds to RCE temperature distribution!

How is $\hat{T}$ distributed?

What are the characteristics of the RCE temperature dist?



A good approximation for the annual-mean solar insolation is:

$$\bar{S}(\phi) = \frac{S_0}{4}\left\{1 - \gamma\left(\sin^2\phi - \frac{2}{3}\right)\right\}$$

So a reasonable form for $\hat{T}$ is

$$\hat{T}^{RCE} = T_0^{RCE} - \Delta T_0^{RCE} \sin^2\phi$$

We then have,

$$\frac{\partial T_0^{RCE}}{\partial \phi} = -2\Delta T_0^{RCE} \sin\phi\cos\phi$$

And our tropopause zonal wind is,

$$\frac{u_t}{\Omega R_e} = \left(\left\{1 + \frac{2R_d \ln\left(\frac{p_s}{p_t}\right)\Delta T_0}{\Omega^2 R_e^2}\right\}^{\frac{1}{2}} - 1\right)\cos\phi$$

for $\phi = 0$

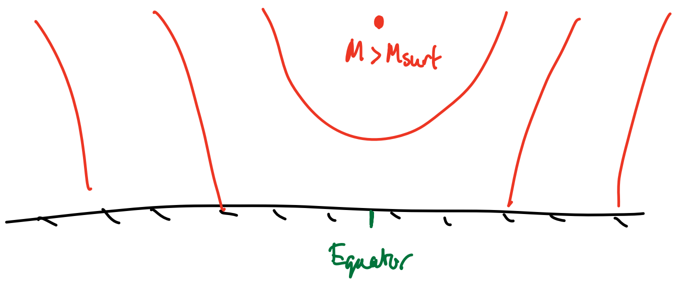
$$\frac{u_t}{\Omega R_e} = \left(\left\{1 + \frac{2R_d \ln\left(\frac{p_s}{p_t}\right)\Delta T_0^{RCE}}{\Omega^2 R_e^2}\right\} - 1\right)$$

$$\frac{u_t}{\Omega R_e} > 0 \quad \text{for} \quad \Delta T_0^{RCE} > 0$$

Note that at $\phi = 0$, $u_t > 0$

Westerlies at the equator!

This implies the tropopause has a higher angular momentum than the surface.



$\Rightarrow$ Hide's theorem is violated if the RCE entropy distribution has non zero curvature at the equator!

Summary

- Thermal wind relates tropopause zonal wind to tropospheric average temperature gradients (assuming $u(z=0)$ is small)

- For reasonable RCE distribution of temperature $(\hat{T})$ winds at equator are westerly

- This violates Hide's theorem, and implies that the steady, inviscid solution is a singular limit.

- This shows the necessity of a circulation!

What does this nearly inviscid circulation look like?

Retaining our axisymmetric assumption, the equation for angular momentum for steady inviscid flow is,

$$\underset{\sim}{u} \cdot \nabla M = 0$$

$\Rightarrow$ velocity vectors must be perpendicular to ∇M

i.e., velocity is parallel to M contours

M is conserved along streamlines!

Next time, we will describe this angular-momentum conserving circulation.