Lecture 8 : Maintanence of a barotropic jet

In the previous lecture, we saw that

- The midlatitude circulation is strongly influenced by the distribution of eddy fluxes of angular momentum
- The convergence of momentum due to eddies is responsible for the thermally indirect Ferrel cell and the surface westerlies

Now we seek to understand what sets the distribution of eddy momentum fluxes on Earth.

Vorticity & Circulation

Consider the vorticity, a quantity related to angular momentum

Vorticity defined:

$$\underset{\sim}{\zeta} = \nabla \times \underset{\sim}{u}$$

Interested in the radial component:

$$\zeta = \underset{\sim}{\zeta} \cdot \underset{\sim}{\hat{r}} = \frac{1}{r\cos\phi}\left\{ \frac{\partial v}{\partial \lambda} - \frac{\partial u\cos\phi}{\partial \phi} \right\}$$

Under the thin shell approx.:

$$\zeta = \frac{1}{R_e\cos\phi}\left\{ \frac{\partial v}{\partial \lambda} - \frac{\partial u\cos\phi}{\partial \phi} \right\}$$

Conservation of vorticity

We begin by considering a single layer homogenous fluid

- constant density: $\rho = \rho_0$
- No vertical variation: $\frac{\partial u}{\partial p} = \frac{\partial v}{\partial p} = 0$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

may be written

$$\rho_0 (\nabla \cdot \underline{u}) = 0$$

$$\Rightarrow \frac{1}{a_e \cos\phi} \left(\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi}(v \cos\phi) \right) = 0$$

Also, can show that the vorticity equation is given by

$$\frac{D(f+\zeta)}{Dt} = (f+\zeta) \nabla_h \cdot \underline{u}$$

Now, restrict ourselves to a constant depth fluid

$$w = 0$$

$$\nabla \cdot \underline{u} = \nabla_h \cdot \underline{u} = 0$$

$$\Rightarrow \frac{D(f+\zeta)}{Dt} = 0$$

Absolute vorticity is conserved!

Derivation of Vorticity equation

We consider inviscid, homogenous ($\rho = \rho_0$) flow of a single layer $\left(\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \right)$

Consider the usual primitive equations:

$$\frac{Du}{Dt} = 2\Omega \sin\phi\, v + \frac{uv \tan\phi}{R_e} - \frac{1}{R_e \rho_0 \cos\phi} \frac{\partial p}{\partial \lambda} + F_\lambda \qquad (1)$$

$$\frac{Dv}{Dt} = -2\Omega \sin\phi\, u - \frac{u^2 \tan\phi}{R_e} - \frac{1}{R_e \rho_0} \frac{\partial p}{\partial \lambda} + F_\phi \qquad (2)$$

Consider the equation for u:

$$\text{LHS} = \frac{\partial u}{\partial t} + \frac{u}{R_e \cos\phi} \frac{\partial u}{\partial \lambda} + \frac{v}{R_e} \frac{\partial u}{\partial \phi}$$

Since for our single layer $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$

Now, note that,

$$v\zeta = \frac{v}{R_e \cos\phi} \left[\frac{\partial v}{\partial \lambda} - \frac{\partial (u\cos\phi)}{\partial \phi} \right]$$

$$= -\frac{v}{R_e} \frac{\partial u}{\partial \phi} + \frac{uv \sin\phi}{R_e \cos\phi} + \frac{1}{R_e \cos\phi} \frac{\partial}{\partial \lambda}\left(\frac{v^2}{2} \right)$$

$$\therefore \quad \frac{v}{R_e} \frac{\partial u}{\partial \phi} = -v\zeta + \frac{uv \tan\phi}{R_e} + \frac{1}{R_e \cos\phi} \frac{\partial}{\partial \lambda}\left(\frac{v^2}{2} \right)$$

We may therefore write,

$$LHS = \frac{\partial u}{\partial t} + \frac{1}{R_e\cos\phi}\frac{\partial}{\partial\lambda}\left\{\frac{u^2+v^2}{2}\right\} - v\varsigma + \frac{uv\tan\phi}{R_e}$$

Putting this back into the u-equation,

$$\frac{\partial u}{\partial t} - v\varsigma + \frac{1}{R_e\cos\phi}\frac{\partial}{\partial\lambda}\left\{\frac{u^2+v^2}{2}\right\} + \frac{uv\tan\phi}{R_e} = 2\Omega\sin\phi\, v + \frac{uv\tan\phi}{R_e} - \frac{1}{R_e\rho_0\cos\phi}\frac{\partial p}{\partial\lambda} + F_\lambda$$

Therefore we have,

$$\frac{\partial u}{\partial t} - v(f+\varsigma) = \frac{-1}{R_e\cos\phi}\frac{\partial}{\partial\lambda}\left\{\frac{u^2+v^2}{2} + \frac{p}{\rho_0}\right\} + F_\lambda \qquad (3)$$

Similarly, the v-equation may be expressed,

$$\frac{\partial v}{\partial t} + u(f+\varsigma) = -\frac{1}{R_e}\frac{\partial}{\partial\phi}\left\{\frac{u^2+v^2}{2} + \frac{p}{\rho_0}\right\} + F_\phi \qquad (4)$$

Now we take $\frac{\partial}{\partial\phi}\left\{(4) - \cos\phi(3)\right\}$

The RHS becomes

$$RHS = \frac{-1}{R_e}\frac{\partial^2}{\partial\phi\partial\lambda}\left(\cancelto{0}{\frac{u^2+v^2}{2}} + \frac{p}{\rho_0}\right) - \frac{1}{R_e}\frac{\partial^2}{\partial\phi\partial\lambda}\left(\cancelto{0}{\frac{u^2+v^2}{2}} + \frac{p}{\rho_0}\right) + \frac{\partial F_\phi}{\partial\phi} - \frac{\partial\cos\phi F_\lambda}{\partial\lambda}$$

On the LHS we have

$$\text{LHS} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial \lambda}\right) + \frac{\partial u}{\partial \lambda}(f+\zeta) + u\frac{\partial}{\partial \lambda}(f+\zeta)$$
$$- \left\{ \frac{\partial}{\partial t}\left(\frac{\partial u\cos\phi}{\partial \phi}\right) - \frac{\partial v\cos\phi}{\partial \phi}(f+\zeta) - v\cos\phi\frac{\partial}{\partial \phi}(f+\zeta) \right\}$$

Rearranging,

$$\text{LHS} = \frac{\partial}{\partial t}\left\{ \frac{\partial v}{\partial \lambda} - \frac{\partial u\cos\phi}{\partial \phi} \right\} + (f+\zeta)\left\{ \frac{\partial u}{\partial \lambda} + \frac{\partial v\cos\phi}{\partial \phi} \right\}$$
$$+ u\frac{\partial(f+\zeta)}{\partial \lambda} + v\cos\phi\frac{\partial(f+\zeta)}{\partial \phi}$$

Dividing by $R_e\cos\phi$, we have,

$$\text{LHS} = \frac{\partial \zeta}{\partial t} + \underset{\sim}{u}\cdot\nabla(\zeta+f) + \nabla\cdot\underset{\sim}{u}(f+\zeta)$$

$$\text{RHS} = \frac{1}{R_e\cos\phi}\left\{ \frac{\partial F_\phi}{\partial \lambda} - \frac{\partial F_\lambda\cos\phi}{\partial \phi} \right\} = \nabla\times\underset{\sim}{F}_h$$

Therefore, we have

$$\frac{D}{Dt}(f+\zeta) = \nabla\times\underset{\sim}{F}_h - (\zeta+f)\nabla\cdot\underset{\sim}{u}_h$$

For single layer non-divergent frictionless flow,

$$\frac{D}{Dt}(\zeta+f) = 0$$

$\Rightarrow$ Absolute vorticity is conserved!

Absolute Vorticity

$$\zeta_a = \underbrace{\zeta}_{\text{relative vorticity}} + \underbrace{f}_{\text{planetary vorticity}}$$

- For a resting fluid ζ_a increases monotonically
- ζ_a is dominated by f in mid- to high latitudes
- $\partial_\phi \zeta_a > 0$ for realistic flows.

Kelvin's Circulation Theorem

Consider the integral of absolute vorticity over some material region of the atmosphere

$$\Gamma(t) = \int_{A(t)} \zeta + f \; dA$$

Now, for an infinitesimal region: $\frac{d\Gamma}{dt} = \frac{d}{dt}\left\{(\zeta + f)\,\delta A\right\}$

But $\frac{d(\zeta+f)}{dt} = 0$ from above, and $\frac{d(\delta A)}{dt} = 0$ by incompressibility

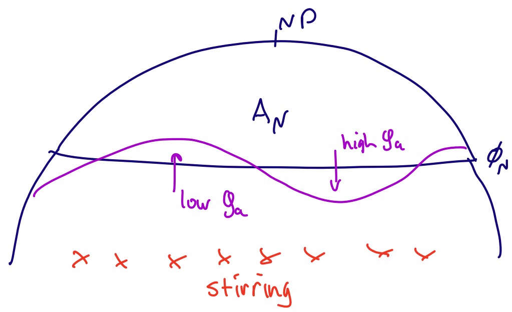
Thus we have a version of Kelvin's circulation theorem: $\frac{d\Gamma}{dt} \overset{!}{=} 0$

Furthermore, by Stokes' theorem, we may write,

$$\Gamma = \int_{A(t)} \zeta + f \; dA = \oint_{\partial A(t)} \underset{\sim}{u}_{abs} \cdot d\underset{\sim}{r}$$

where ∂A is the boundary of A, $\underset{\sim}{u}_{abs}$ is the absolute zonal velocity

Lets apply Kelvin's circulation theorem to a "polar cap" up to a latitude ϕ_0



$$\Gamma_{\phi_0} = \int_A \zeta + f \, dA = \int_{\partial A} \underline{u} \cdot d\underline{r} = 2\pi R_e \cos\phi \, [u]$$

The mean zonal flow is equal to the sum of the vorticity further polewards!

Now let us stir the fluid at some latitude south of ϕ_N. As the disturbance reaches ϕ_N, the contours will deform.

Since $\partial_\phi \zeta_a > 0$, this advects low vorticity air into the polar cap, and advects high vorticity air away.

Redo our calculation of circulation and we have,

$$\Gamma^f_{\phi_N} < \Gamma^i_{\phi_N} \quad \Rightarrow \quad [u]^f_{\phi_N} < [u]^i_{\phi_N}$$

The stirring thus produces a deceleration of the flow on its poleward flank.

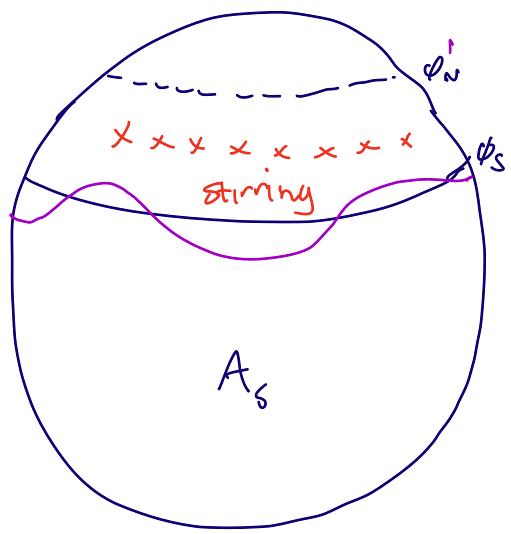
Some time later, we might imagine the flow returns to its original position reversibly. In this case the velocity would revert to its initial value.

However, if there is some irreversible mixing that occurs, we might expect a net flux of vorticity through the latitude circle.

Summary:

Stirring produced south of ϕ_0 can act to decelerate the flow in the polar cap.

A similar argument may be applied to the region south of the stirring:



Deformation of contour originally at ϕ_S advects low vorticity northwards. But note that the surface is now oriented in the opposite sense!

$$\Gamma_{\phi_S} = -\int_{A_S} (\zeta + f)\, dA_S = R_e \cos\phi\, [u]_{\phi_S}$$

$$\Gamma^f_{\phi_S} < \Gamma^i_{\phi_S} \quad \Rightarrow \quad [u]^f_{\phi_S} < [u]^i_{\phi_S}$$

The stirring decelerates the flow to the south!

We have shown that stirring that produces disturbances that propagate away and decay irreversibly produces decceleration in the regions of decay. By conservation of angular momentum, the zonal wind in the source region must accelerate westerly!

Westerlies form in the stirred region!

But how is this momentum transport effected?

Rossby waves and momentum flux

Let us consider the momentum transport produced by propagating Rossby waves.

For simplicity stick with our non-divergent barotropic flow

$$\frac{D(\zeta+f)}{Dt}=0$$

Consider β-plane approximation $(\lambda,\phi)\rightarrow(x,y)$, $f=f_0+\beta y$

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$$ (no vertical velocity)

$$\frac{Df}{Dt}=\beta v$$

$$\Rightarrow \quad \frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+\beta v=0$$

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\left(\frac{\partial \zeta}{\partial y}+\beta\right)=0 \quad , \quad \beta=\frac{df}{dy}$$

Linearise about a basic state $\underset{\frown}{u} = (\bar{u}, 0)$

where we assume $\bar{u}$ is uniform:

$$\frac{\partial \zeta}{\partial t} + \bar{u}\frac{\partial \zeta}{\partial x} + \beta v' = 0$$

Define a streamfunction φ so that

$$u' = -\frac{\partial \varphi}{\partial y}, \quad v' = \frac{\partial \varphi}{\partial x}$$

$$\zeta \stackrel{<}{=} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \nabla^2 \varphi$$

Search for solutions of the form:

$$\varphi = A\,\mathrm{Re}\left\{e^{i(kx+ly-\omega t)}\right\}, \quad \text{where } A \text{ is wave amplitude}$$

Note that for this plane wave form,

$$u'v' = A^2\,\mathrm{Re}\left\{-il\exp\left[i(kx+ly-\omega t)\right]\right\}\mathrm{Re}\left\{ik\exp\left[i(kx+ly-\omega t)\right]\right\}$$

$$= -A^2 kl\sin^2(kx+ly-\omega t)$$

Taking the time average over one period:

$$\overline{u'v'} = \frac{-A^2 k l}{2}$$

Now, let us calculate the dispersion relation.

We have,

$$-(k^2+l^2)(-\omega + k\bar{u}) + k\beta = 0$$

$$\Rightarrow \quad \omega = k\bar{u} - \frac{k\beta}{(k^2+l^2)}$$

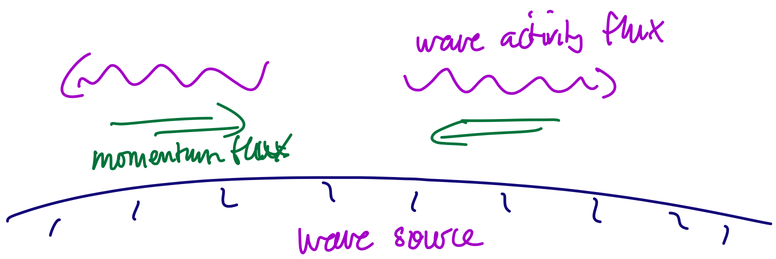
The meridional group velocity is then given by

$$C_{gy} = \frac{\partial \omega}{\partial l} = 2l \cdot \frac{k\beta}{(k^2+l^2)^2} = kl \underbrace{\left(\frac{2\beta}{(k^2+l^2)^2}\right)}_{\text{Positive definite}}$$

The momentum flux is proportional to, but opposite in sign to the meridional group velocity!

Remember that the group velocity tells us about the propogation of energy by the waves (actually wave activity)

So the waves flux momentum in the opposite direction to wave activity

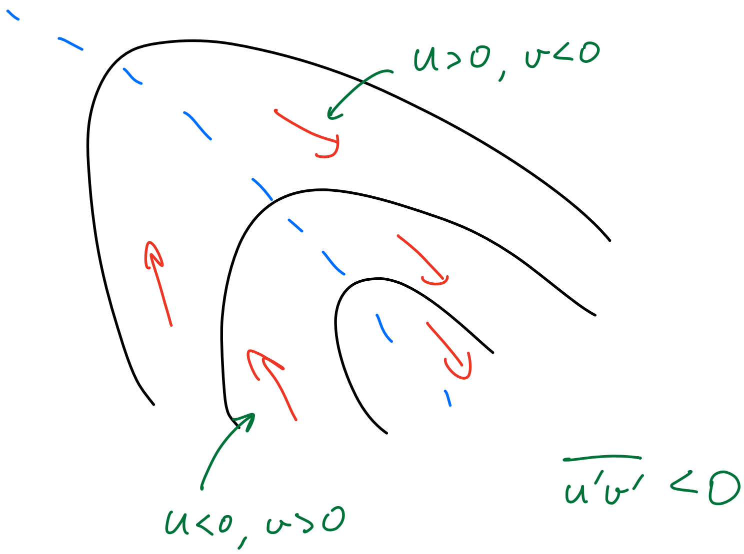


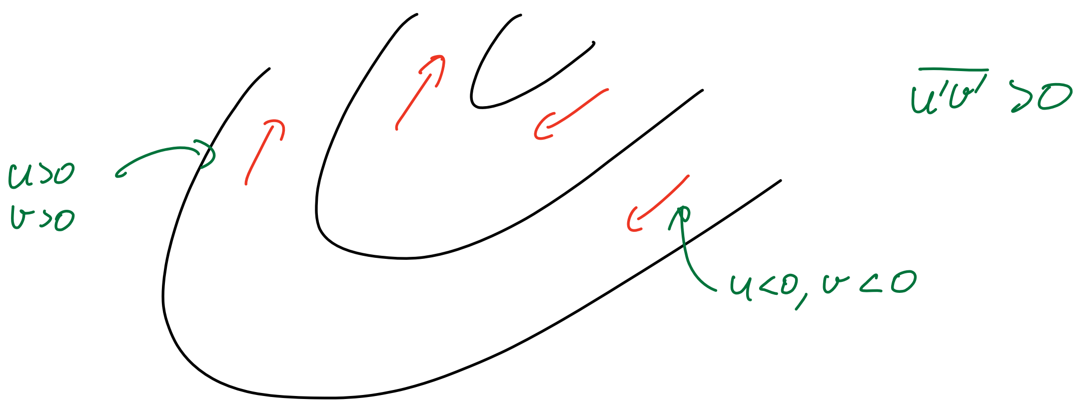
As before, momentum is converged into the stirred region!

Kinematics of momentum transport

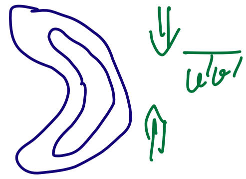
What does this momentum flux look like?

$\overline{u'v'} \neq 0$ can be achieved by "tilted" troughs





Convergence of momentum can be achieved by "Banana-shaped" eddies



Propagation of Rossby waves

Consider again the dispersion relation

$$\omega = k\bar{u} - \frac{\beta}{(k^2+l^2)}$$

For a given frequency and zonal wavelength, we can solve for l:

$$(k\bar{u} - \omega)(k^2+l^2) = k\beta$$

$$l^2 = \frac{k\beta}{k\bar{u}-\omega} - k^2$$

$$l = \pm\left(\frac{k\beta}{\bar{u}-c} - k^2\right)^{\frac{1}{2}}$$

where $c = \frac{\omega}{k}$ is the zonal phase speed.

$\Rightarrow$ propagation is allowed ($l \in \mathbb{R}$) provided

$$\frac{\beta}{\bar{u}-c} > k^2$$

This requires that $\bar{u} - c > 0$

$\Rightarrow$ The Doppler shifted phase speed must be to the west

- The waves must be moving west relative to the mean flow.

We have derived this assuming $\bar{u}$ is constant.

But similar results apply even if $\bar{u}$ varies, as long as we can apply a WKB approximation.

In this case we may write an approximate solution

$$\psi(x,y,t) = A(y) \exp\{ i(kx + \tilde{l}y - \omega t\}$$

where l is the local solution to

$$\tilde{l}^2(y) = \frac{\tilde{\beta}}{[\bar{u}] - c} - k^2$$

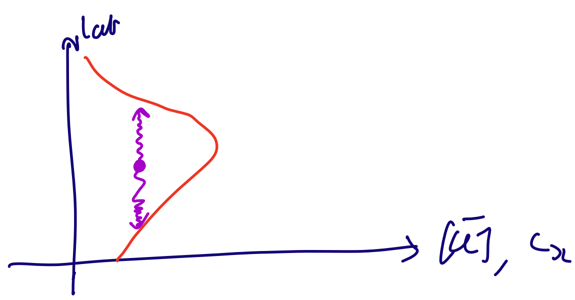
$$\tilde{\beta} = \frac{\partial (f + [\bar{q}])}{\partial y}$$

$c = \dfrac{\omega}{k}$ is conserved!

(slightly different in spherical geometry)

As $[\bar{u}] - c \to 0$

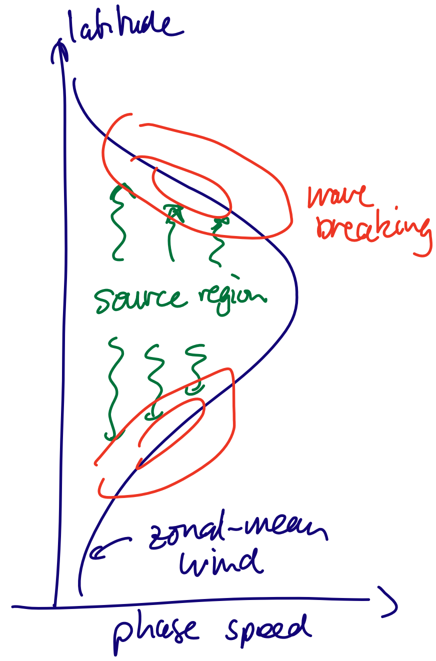
- group velocity slows ($c_g \to 0$)
- energy density increases
- Linear assumption breaks down



- The location where $[\bar{u}] - c = 0$, $l \to \infty$, $c_g \to 0$ } critical latitude
  $\Rightarrow$ this is where the wave breaks.

- where $l \to 0$ is a turning latitude $\Rightarrow$ wave energy is reflected.

Jet maintenance - conceptual picture



- Waves produced by baroclinic instability at midlatitudes
- Waves propagate meridionally and approach critical latitude where
  $$\bar{u} - c \approx 0$$
- Waves break and dissipate near critical latitude
- Momentum is transported to centre of jet

Epilogue

- We have come a long wave with a simple single layer barotropic model of the atmosphere.
- In single-layer case eddy stirring must be imposed
- In reality eddies at midlatitudes a result of baroclinic instability
- This requires at least 2-layer model to understand
- See Vallis (2006) ch. 12, Held notes

Still need to understand what sets the eddy field and how this interacts with the mean flow. Will return to this point later with the powerful technique of the transformed Eulerian mean.