Lecture 10 : Eliassen - Palm fluxes

Last lecture:

- Showed that EP flux divergence represents net effect of eddies on $[u]$.
- This allowed definition of TEM formulation
- Now connect EP flux to dynamics of potential vorticity

Quasi-geostrophic potential vorticity:

Consider again the β-plane equations under QG scaling.

$$\frac{\partial u_g}{\partial t} + u_g\frac{\partial u_g}{\partial x} + v_g\frac{\partial u_g}{\partial y} = fv - \frac{\partial \Phi}{\partial x} + \cancel{F_x} \qquad (a)$$

$$\frac{\partial v_g}{\partial t} + u_g\frac{\partial v_g}{\partial x} + v_g\frac{\partial v_g}{\partial y} = -fu - \frac{\partial \Phi}{\partial y} + \cancel{F_y} \qquad (b)$$

$$\frac{\partial \theta}{\partial t} + u_g\frac{\partial \theta}{\partial x} + v_g\frac{\partial \theta}{\partial y} + \omega\sigma = \cancel{\frac{\dot{q}}{c_p\pi}} \qquad (c)$$

Consider adiabatic frictionless flow

Take $\frac{\partial}{\partial x}(b) - \frac{\partial}{\partial y}(a)$

$$LHS = \frac{\partial}{\partial t}(\zeta_g) + u_g\frac{\partial \zeta_g}{\partial x} + v_g\frac{\partial \zeta_g}{\partial y} + \overbrace{\cancel{\frac{\partial u_g}{\partial x}\frac{\partial v_g}{\partial x}} + \cancel{\frac{\partial v_g}{\partial x}\frac{\partial v_g}{\partial y}}}^{0} - \overbrace{\cancel{\frac{\partial u_g}{\partial y}\frac{\partial u_g}{\partial x}} - \cancel{\frac{\partial v_g}{\partial y}\frac{\partial u_g}{\partial y}}}^{0}$$

$$LHS = \frac{D_g}{Dt}(\zeta_g)$$

$$RHS = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - v\frac{df}{dy}$$

Now, under QG scaling

$$-f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -f_0\frac{\partial \omega}{\partial p}$$

Equation (c) gives:

$$\omega = -\frac{1}{\sigma}\left\{\frac{\partial\theta}{\partial t} + u_g\frac{\partial\theta}{\partial x} + v_g\frac{\partial\theta}{\partial y}\right\}$$

$$\Rightarrow \quad -f_0\frac{\partial\omega}{\partial p} = \frac{D_g}{Dt}\left\{f_0\frac{\partial}{\partial p}\left(\frac{\theta}{\sigma}\right)\right\}$$

Hence we have,

$$\frac{D_g}{Dt}(\zeta_g) = -v\frac{df}{dy} + \frac{D_g}{Dt}\left\{\frac{f_0}{\sigma}\frac{\partial\theta}{\partial p}\right\}$$

$$\frac{D_g}{Dt}\left\{\zeta_g + f - \frac{f_0}{\sigma}\frac{\partial\theta}{\partial p}\right\} = 0$$

Conservation condition

$q = \zeta_g + f - \frac{f_0}{\sigma}\frac{\partial\theta}{\partial p}$ is the QGPV

QGPV is conserved following the geostrophic flow!

QGPV may also be expressed entirely as a function of the geopotential:

$$q = \beta y + \frac{1}{f_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi + \frac{\partial}{\partial p}\left(\frac{f_0}{S}\frac{\partial\Phi}{\partial p}\right)$$

invertibility condition

where $S = \frac{\sigma R\pi}{p}$

and $u_g = \frac{1}{f_0}\frac{\partial\Phi}{\partial y}$, $v_g = -\frac{1}{f_0}\frac{\partial\Phi}{\partial x}$, $\frac{\theta}{\sigma} = \frac{-1}{S}\frac{\partial\Phi}{\partial p}$

Now take zonal mean, remembering that $[v_g] = 0$

$$\frac{\partial[q]}{\partial t} + \frac{\partial}{\partial y}[v_g q] = 0$$

$$\Rightarrow \quad \frac{\partial[q]}{\partial t} = -\frac{\partial}{\partial y}[v_g^* q^*]$$

What is this eddy QGPV flux?

$$v^* q^* = v^* \left\{ \frac{\partial v^*}{\partial x} - \frac{\partial u^*}{\partial y} + f_0 \frac{\partial}{\partial p}\left(\frac{\theta^*}{\sigma}\right) \right\}$$

$$= v^* \frac{\partial v^*}{\partial x} - v^* \frac{\partial u^*}{\partial y} + v^* f_0 \frac{\partial}{\partial p}\left(\frac{\theta^*}{\sigma}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2} v^{*2}\right) - v^* \frac{\partial u^*}{\partial y} \underbrace{- u^* \frac{\partial v^*}{\partial y} - u^* \frac{\partial u^*}{\partial x}}_{0 \text{ by cty}} + v^* f_0 \frac{\partial}{\partial p}\left(\frac{\theta^*}{\sigma}\right)$$

$$= \frac{\partial}{\partial x}\left\{\frac{1}{2}\left(v_g^{*2} - u_g^{*2}\right)\right\} - \frac{\partial (u_g^* v_g^*)}{\partial y} + v_g^* f_0 \frac{\partial}{\partial p}\frac{\theta^*}{\sigma} + \frac{\theta^*}{\sigma} f_0 \frac{\partial v_g^*}{\partial p} - \frac{\theta^*}{\sigma} f_0 \frac{\partial v_g^*}{\partial p}$$

Now $f_0 \frac{\partial v_g^*}{\partial p} = \frac{-R\pi}{p} \frac{\partial \theta^*}{\partial x}$ by thermal wind

$$\Rightarrow \quad v^* q^* = \frac{\partial}{\partial x}\left\{\frac{1}{2}\left(v_g^{*2} - u_g^{*2} - \frac{R\pi}{\sigma p}\theta^{*2}\right)\right\} - \frac{\partial}{\partial y} u_g^* v_g^* + f_0 \frac{\partial}{\partial p}\left(\frac{v_g^* \theta^*}{\sigma}\right)$$

$$[v^* q^*] = -\frac{\partial}{\partial y}[u_g^* v_g^*] + f_0 \frac{\partial}{\partial p}\left(\frac{v_g^* \theta^*}{\sigma}\right)$$

$$= \nabla \cdot \underset{\sim}{F}$$

The QGPV flux is proportional to the divergence of the E-P flux!

So the zonal momentum equation is

$$\frac{\partial [u_g]}{\partial t} - f_0 [\tilde{v}] = [v^* q^*] + [F_x]$$

- Total explicit forcing is by the eddy PV flux
- Advection of heat is by the residual mean circulation

PV-thinking & the general circulation

- QGPV provides an efficient description of circulation (invertibility)

- QGPV conserved → can show eddy fluxes must be downgradient (on average)
  → Cf. momentum fluxes

- Simple model of extratropical circulation can be constructed based on diffusion of PV down the gradient

But if we diffuse PV, how do we get jets?

Need one more element!

Baroclinic Rossby waves

Let's consider Rossby waves in our QG system.

Governing equation: $$\frac{\partial q}{\partial t} + u_g \frac{\partial q}{\partial x} + v_g \frac{\partial q}{\partial y} = 0$$

for adiabatic, inviscid flow.

Linearise about a mean zonal flow $[u]$ with PV gradient $[q]_y = \beta$

$$\frac{\partial q'}{\partial t} + [u]\frac{\partial q'}{\partial x} + v'\beta = 0 \qquad (1)$$

We have that $$q' = \frac{1}{f_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi' + \frac{\partial}{\partial p}\frac{f_0}{S}\frac{\partial \Phi}{\partial p}$$

and $$v' = \frac{1}{f_0}\frac{\partial \Phi}{\partial x}$$

Taking a wave-like form: $\Phi' = Ae^{i(kx+ly+mp-\omega t)}$, assume S is constant:

$$q' = \left[\frac{1}{f_0}\left(k^2+l^2\right) + \frac{f_0}{S}m^2\right]\Phi'$$

$$v' = \frac{1}{f_0}k\Phi$$

Substitute into (1):

$$\left(\omega+[u]k\right)\left\{\frac{1}{f_0}\left(k^2+\ell^2\right)+\frac{f_0}{S}m^2\right\}+\frac{1}{f_0}k\beta=0$$

$$\left(\omega+[u]k\right)\underbrace{\left[k^2+\ell^2+\frac{f_0^2}{S}m^2\right]}_{K^2}+k\beta=0$$

$$\left(\omega+[u]k\right)K^2+k\beta=0$$

$$\omega=[u]k-\frac{k\beta}{K^2}$$

$$\frac{\omega}{k}=c_x=[u]-\frac{\beta}{K}$$

Very similar to the barotropic case:

$$c_x=[u]-\frac{\beta}{k^2+\ell^2} \qquad \text{(Barotropic)}$$

$$c_x=[u]-\frac{\beta}{k^2+\ell^2+\frac{f_0^2}{S}m^2} \qquad \text{(Baroclinic)}$$

Also note that in general β can be replaced by

$$[q]_y=\frac{\partial}{\partial y}\left\{\zeta_g+f-f_0\frac{\partial}{\partial p}\left(\frac{\theta}{\sigma}\right)\right\}$$

For the case where $[v]=0$, $\frac{\partial[\theta]}{\partial p}=0$

$$\Rightarrow [q]_y=\beta-[u]_{yy}$$

Rossby waves & EP fluxes

Remember in our barotropic discussion we used the vorticity equation to derive the dispersion relation for barotropic Rossby waves:

$$\omega = [u]k - \frac{\beta k}{k^2+l^2}$$

We can follow a similar procedure for the QGPV equation to derive the dispersion relation for baroclinic Rossby waves

$$\omega = [u]k - \frac{\beta k}{K^2} \quad , \quad K^2 = k^2+l^2+m^2\frac{f_0^2}{S}$$

for a solution of the form $\Phi = A\cos(kx+ly+mp-\omega t)$

Calculate the group velocity: (see problem set 4)

$$C_{gy} = \frac{\partial \omega}{\partial l} = \frac{2\beta k l}{K^4}$$

$$C_{gp} = \frac{\partial \omega}{\partial m} = \frac{2\beta k m}{K^4}\left(\frac{f_0^2}{S}\right)$$

Now, $u_g^* = \frac{1}{f_0}\frac{\partial \Phi}{\partial y} = \frac{-Al}{f_0}\sin(kx+ly+mp-\omega t)$

$$v_g^* = \frac{-1}{f_0}\frac{\partial \Phi}{\partial y} = +\frac{Ak}{f_0}\sin(kx+ly+mp-\omega t)$$

$$\frac{\theta^*}{\sigma} = \frac{-1}{S}\frac{\partial \Phi}{\partial p} = +\frac{Am}{S}\sin(kx+ly+mp-\omega t)$$

$$-[u_g^* v_g^*] = \int_0^{2\pi/k} +\frac{A^2 lk}{f_0^2}\sin^2(kx+ly+mp-\omega t)\, dx = \frac{A^2 lk}{f_0^2}$$

$$f_0\frac{[v_g^*\theta^*]}{\sigma} = \int_0^{2\pi/k} \frac{f_0 A^2 km}{S f_0}\sin^2(kx+ly+mp-\omega t)\, dx = \frac{A^2}{S}km$$

Group velocity given by:

$$\underset{\sim}{c_g} = (c_{gy}, c_{gp}) = \frac{2\beta}{q\kappa^4}\left(kl, \ km\frac{f_0^2}{S}\right)$$

EP flux given by:

$$\underset{\sim}{F} = \left(-[u_g^* v_g^*], \ f_0\frac{[v_g^*\theta^*]}{\sigma}\right)$$

$$= \frac{A^2}{f_0^2}\left(kl, \ \frac{f_0^2}{S}km\right)$$

$\underset{\sim}{c_g} \parallel \underset{\sim}{F}$

EP flux vectors show the direction of propagation of wave "energy"

Indeed, can construct a wave activity equation:

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla\cdot(\underset{\sim}{c_g}\mathcal{A}) = 0$$

where $\underset{\sim}{F} = \underset{\sim}{c_g}\mathcal{A}$

and $\mathcal{A} = \frac{1}{2}\frac{[q^{*2}]}{[q_y]}$ is the wave activity

But remember the non-acceleration result:

if $\nabla\cdot\underset{\sim}{F} = 0$ a steady, adiabatic, non dissipative

solution is $[u_g] = [\theta] = [\tilde{\omega}] = [\tilde{v}] = 0$

A steady, propagating, wave has $\nabla\cdot(\underset{\sim}{c_g}\mathcal{A}) = 0$

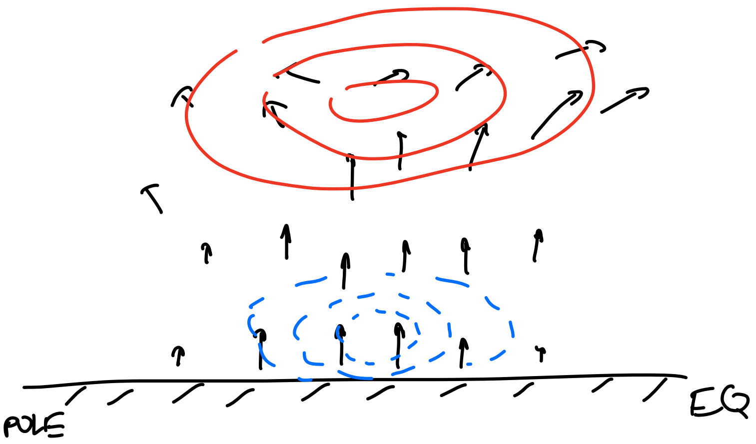
$\Rightarrow \nabla\cdot\underset{\sim}{F} = 0$, no acceleration

Waves only affect mean flow at their source $\left(\nabla\cdot(\underset{\sim}{c_g}\mathcal{A}) > 0\right)$

or where they are absorbed $\left(\nabla\cdot(\underset{\sim}{c_g}\mathcal{A}) < 0\right)$

(by breaking or otherwise)

E-P Fluxes



- Waves produced at low levels at midlatitudes
- propagate upward, turn equatorward (mostly) at upper levels
- EP flux convergence at low levels, divergence at high levels

- Associated with poleward heat flux
- convergence of momentum at upper levels

Jet formation

We have shown that the EP flux may be interpreted in 3 ways:

1) EP flux divergence is equal to the total forcing of the zonal wind

2) EP flux divergence is equal to the meridional flux of QGPV

3) EP fluxes are parallel to the flux of wave activity

=> Use "Rossby wave chromatography" and assume where waves break QGPV is mixed.

→ Consider uniform vorticity gradient

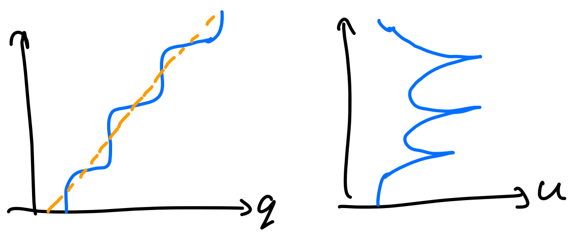
→ Suppose an eddy mixes a particular region

→ Now the edges of the mixed regions have high QGPV gradient so Rossby waves propagate well there

→ In the mixed region $\partial_y q$ is small → waves break, mixing further

→ End up with a PV staircase of mixed regions and sharp jets

→ What sets the jet spacing?



Summary

- In extratropics, temperature & zonal wind strongly coupled by thermal wind balance
- Eddy fluxes induce overturning circulations that affect both [u] and [θ]
- Must therefore consider eddy fluxes of heat + momentum together
- Efficient way to do this is through Transformed Eulerian Mean
- Define "residual circulation", that takes into account heat fluxes by eddies
- Residual overturning gives "diabatic" circulation (similar to circulation in isentropic coordinates)
- Thermally direct at all latitudes
- TEM formulation includes eddy effects within EP fluxes
- EP flux divergence important because
  - represents eddy forcing on zonal mean
  - represents QGPV flux
  - represents source/sink wave activity