Ass 1 solutions

Problem 1

(a) $$\frac{Du}{Dt} = 2\Omega \sin\phi\, v + \frac{uv}{R_e}\tan\phi - \frac{1}{R_e\cos\phi}\frac{\partial\Phi}{\partial\lambda} + F_\lambda \quad \cdots (1)$$

Note that $$\frac{D(R_e\cos\phi)}{Dt} = -R_e\sin\phi\frac{D\phi}{Dt} \quad \text{by chain rule} \quad \cdots (2)$$

$$= -\sin\phi\, v \quad \text{since} \quad \frac{D\phi}{Dt} = \frac{v}{R_e}$$

and $$\frac{D}{Dt}\left\{R_e^2\cos^2\phi\right\} = -2R_e^2\cos\phi\sin\phi\frac{D\phi}{Dt} \quad \cdots (3)$$

$$= -2R_e\cos\phi\sin\phi\, v$$

Multiply (1) by $R_e\cos\phi$:

$$R_e\cos\phi\frac{Du}{Dt} = 2R_e\Omega\sin\phi\cos\phi\, v + uv\sin\phi - \frac{\partial\Phi}{\partial\lambda} + R_e\cos\phi F_\lambda$$

Using (2) & (3)

$$R_e\cos\phi\frac{Du}{Dt} = \underbrace{-\Omega\frac{D}{Dt}\left\{R_e^2\cos^2\phi\right\}}_{\text{by (3)}} - \underbrace{u\frac{D(R_e\cos\phi)}{Dt}}_{\text{by (2)}} - \frac{\partial\Phi}{\partial\lambda} + R_e\cos\phi F_\lambda$$

$$\underbrace{R_e\cos\phi\frac{Du}{Dt} + u\frac{D(R_e\cos\phi)}{Dt}}_{\text{product rule}} + \frac{D}{Dt}\left\{\Omega R_e^2\cos^2\phi\right\} = -\frac{\partial\Phi}{\partial\lambda} + R_e\cos\phi F_\lambda$$

$$\frac{D}{Dt}\left\{R_e\cos\phi\, u + \Omega R_e^2\cos^2\phi\right\} = -\frac{\partial\Phi}{\partial\lambda} + R_e\cos\phi F_\lambda$$

$$\frac{DM}{Dt} = -\frac{\partial\Phi}{\partial\lambda} + R_e\cos\phi F_\lambda$$

For axisymmetric flow: $\frac{\partial \Phi}{\partial \lambda} = 0$

For inviscid flow: $F_\lambda = 0$

$$\frac{DM}{Dt} = 0$$

(b) We have used the shallow fluid approximation to derive the Primitive Equations. This approximation neglects changes in r relative to R_e. To be consistent with this approximation we must do the same in defining M. Thus we replace r with R_e in the definition of M.

$$M_{full} = r\cos\phi\,(u + \Omega r\cos\phi)$$

$$M_{primitive} = R_e\cos\phi\,(u + \Omega R_e\cos\phi)$$

(c) Flow on Jupiter is not axisymmetric, so Hide's theorem does not apply.

When eddies are considered, the circulation is able to transport angular momentum up the mean gradient accounting for an equatorial jet.

2. (a) RCE corresponds to a state in which convective heating balances radiative cooling locally within each column of the atmosphere.

$\Rightarrow$ There is no large-scale circulation

(b) RCE solution not valid because:

- It is unstable (baroclinically, barotropically, symmetrically)
- It violates Hide's theorem
- Boundary condition is not axisymmetric, precluding axisymmetric solution

(c) RCE zonal velocity at tropopause:

$$\frac{u_t}{\Omega R_e} = \left\{ \left(1 - \frac{R_d \ln\left(\frac{p_s}{p_t}\right)}{\Omega^2 R_e^2 \sin\phi \cos\phi} \frac{\partial \hat{T}}{\partial \phi} \right)^{\frac{1}{2}} - 1 \right\} \cos\phi$$

where $\hat{T} = \frac{1}{\ln\left(\frac{p_s}{p_t}\right)} \int_{p_t}^{p_s} T \, d\ln(p)$

Suppose we have $\hat{T} = T_0 - \delta T \exp\left(\frac{-\phi^2}{2(\delta\phi)^2} \right)$

$$\frac{\partial \hat{T}}{\partial \phi} = -\frac{\delta T}{(\delta\phi)^2} \phi \exp\left(\frac{-\phi^2}{2(\delta\phi)^2} \right)$$

$$\frac{u_t}{\Omega R_e} = \left\{ \left[1 + \frac{R_d \ln\left(\frac{p_s}{p_t}\right)}{\Omega^2 R_e^2 \sin\phi \cos\phi} \frac{\delta T}{(\delta\phi)^2} \phi \exp\left(\frac{-\phi^2}{2(\delta\phi)^2} \right) \right]^{\frac{1}{2}} - 1 \right\} \cos\phi$$

Set $p_s = 1000$ hPa, $p_t = 100$ hPa, $\delta\phi = 30° = \frac{\pi}{6}$, $\delta T = 100$ K

see attached plot.

(d) Solution violates Hide's theorem if it produces a maximum of angular momentum above the surface.

Consider the angular momentum at the tropopause: M_t

A sufficient condition for an angular momentum maximum above the surface is if $M_t(\phi) > \Omega R_e^2$ for some ϕ. Here ΩR_e^2 is the angular momentum of the Earth at the Equator.

In particular, if $u_t(\phi=0) > 0$, then $M_t(\phi=0) > \Omega R_e^2$ and Hide's theorem is violated.

$$\lim_{\phi \to 0} \frac{u_E}{\Omega R_e} = \lim_{\phi \to 0} \left\{ \left(1 + \frac{R d}{\Omega^2 R_e^2} \frac{\delta T}{(\delta \phi)^2} \frac{\phi}{\sin\phi} \right)^{\frac{1}{2}} - 1 \right\}$$

Since $\lim_{\phi \to 0} \frac{\phi}{\sin\phi} = 1$ (l'Hôpital's rule)

$$\Rightarrow \quad \lim_{\phi \to 0} \frac{u_E}{\Omega R_e} = \left\{ \left(1 + \frac{R d}{\Omega^2 R_e^2} \frac{\delta T}{(\delta \phi)^2} \right)^{\frac{1}{2}} - 1 \right\}$$

$\lim_{\phi \to 0} \frac{u_E}{\Omega R_e} > 0$ for any $\delta T > 0$

$\Rightarrow$ Hide's theorem is violated.

Problem 3.

(a) Angular momentum of the solid Earth

$$M_E = \oint_{V_{Earth}} \rho_e r^2 \cos^2\phi \, \Omega \, dV$$

Express in spherical coords assuming $\rho_e = $ const

$$M_E = \rho_e \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{R_e} r^2 \cos^2\phi \, \Omega \, (r^2 \cos\phi) \, dr \, d\phi \, d\lambda$$

$$= \rho_e (2\pi) \Omega \int_0^{R_e} r^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\phi \, d\phi \, dr$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\phi \, d\phi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2\phi) \cos\phi \, d\phi$$

let $v = \sin\phi$, $dv = \cos\phi \, d\phi$

$$I = \int_{-1}^{1} (1 - v^2) \, dv = \left[v - \frac{v^3}{3} \right]_{-1}^{1} = 4/3$$

$$M_E = 2\pi \rho_e \Omega \left(\frac{4}{3}\right) \int_0^{R_e} r^4 \, dr$$

$$= \frac{4}{3}\pi \rho_e \Omega \cdot 2 \cdot \frac{R_e^5}{5}$$

$$= \left(\frac{4}{3}\pi R_e^3 \rho_e\right) \cdot \frac{2R_e^2}{5}\Omega$$

Mass of Earth, $m_E = \rho_e$ (Volume of Earth)

$$= \rho_e \left(\frac{4}{3}\pi R_e^3\right)$$

Hence, $$M_E = m_E \left(\frac{2R_e^2}{5}\Omega\right)$$

(b) Angular momentum of atmosphere:

$$M_{atm} = \oint_{V_{atm}} \rho R_e \cos\phi \left(\Omega R_e \cos\phi + u\right) dV$$

For uniform change in zonal wind Δu

$$\Delta M_{atm} = \oint_{V_{atm}} \rho R_e \cos\phi \, \Delta u \, dV$$

$$= \Delta u \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R_e \cos\phi \left(R_e^2 \cos\phi\right) \left(\int_0^\infty \rho \, dz\right) d\phi \, d\lambda$$

where we have used the thin shell approximation to write the Jacobian $r^2\cos\phi$ as $R_e^2 \cos\phi$, and we have replaced the integral in r by an integral in z.

Now, by hydrostatic balance

$$\int_0^\infty \rho \, dz = \int_0^{p_s} \frac{dp}{g} = \frac{p_s}{g}$$

$$\Delta M_{atm} = \frac{\Delta u p_s}{g} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R_e^3 \cos^2\phi \, d\phi \, d\lambda$$

$$= \frac{\Delta u p_s}{g} \left(2\pi R_e^3\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\phi \, d\phi$$

Note that $\cos(2\phi) = 2\cos^2\phi - 1$ [Google]

$$\cos^2\phi = \frac{\cos(2\phi)+1}{2}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\phi \, d\phi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(2\phi)+1}{2} \, d\phi$$

$$= \frac{\pi}{2}$$

$$\Delta M_{atm} = \frac{\Delta u p_s}{g} \pi^2 R_e^3$$

(c) We have $\frac{dM_{atm}}{dt} = \frac{-d M_{Earth}}{dt}$

Integrating in time

$$\Delta M_{atm} = -\Delta M_{Earth}$$

$\Rightarrow$ change in Earths angular momentum is

$$\Delta M_{Earth} = -\frac{\Delta u p_s}{g} \pi^2 R_e^3$$

Since $M_{Earth} = m_E \left(\frac{2 R_e^2}{5}\right) \Omega$

and mass and radius of Earth do not change,

we have $\quad \Delta M_{Earth} = m_E\left(\frac{2R_e^2}{5}\right)\Delta\Omega$

Also $\quad \frac{\Delta M_{Earth}}{M_{Earth}} = \frac{\Delta\Omega}{\Omega}$

$$\Rightarrow \quad \frac{\Delta\Omega}{\Omega} = \frac{-\frac{1}{g}\Delta u\, P_s\, \pi^2 R_e^3}{m_E\left(\frac{2R_e^2}{5}\right)}$$

$$\frac{\Delta\Omega}{\Omega} = \frac{2\pi^2 P_s R_e}{5 g m_E}\Delta u$$

Finally, length of (Sidereal) day T is given by

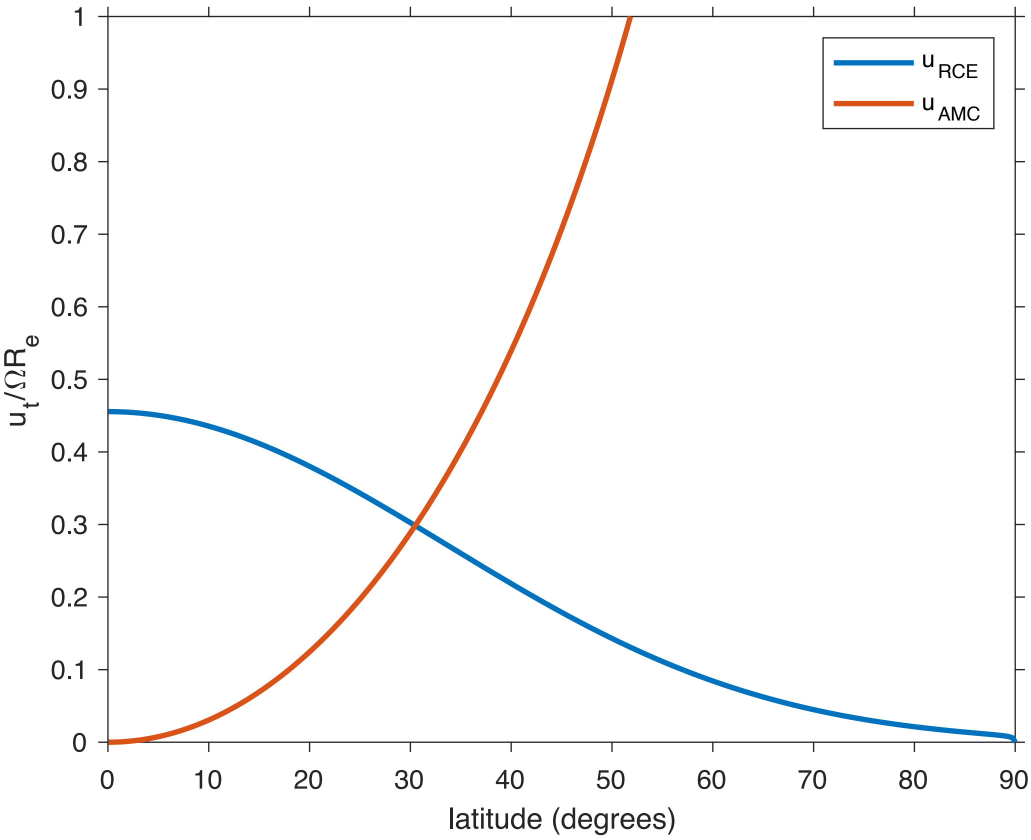
$$T = \frac{2\pi}{\Omega}$$

$$\frac{\Delta T}{T} \simeq -\frac{\Delta\Omega}{\Omega}$$

$$\frac{\Delta T}{T} = \frac{2\pi^2 P_s R_e}{5 g M_E}\Delta u$$

Plugging in numbers, get $\Delta T \sim 0.3$ ms.

Day <u>lengthens</u> by 0.3 ms due to change in u.



Tropopause level (100 hPa) zonal wind non-dimensionalised by the absolute velocity of the surface of the Earth at the equator. Blue line shows RCE solution and red line shows angular-momentum conserving solution.