Lecture 2 : Governing Equations

Basic Equations

Need to solve for 6 key variables

u, v, w
T
p
ρ

Also moisture (q) or salinity (s)

Need 6 equations (+ eq^n for humidity/salinity)

- Mass continuity
- Newton's second law (3 eq^{n}s)
- thermodynamic equation
- equation of state

Equation of State

- Connects temperature, pressure and density
- Involves humidity (salinity in the ocean)

- For atmospheric science, ideal gas law is sufficient

$$\boxed{p = \rho R_d T} \qquad (A)$$

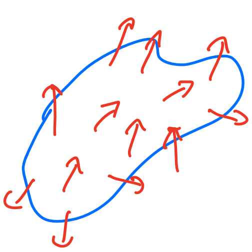
$R_d = 287 \quad J\,kg^{-1}K^{-1}$

What are the assumptions that lead to this?

Conservation of mass

Consider a control volume

Change in mass within volume equals flux of mass through boundary



$$\frac{d}{dt}\oint_V \rho \, dV = -\oint_{\partial V} \rho \underset{\sim}{u} \cdot dS$$

By divergence theorem:

$$\oint_V \frac{\partial \rho}{\partial t}\, dV = -\oint_V \nabla \cdot \rho \underset{\sim}{u}\, dV$$

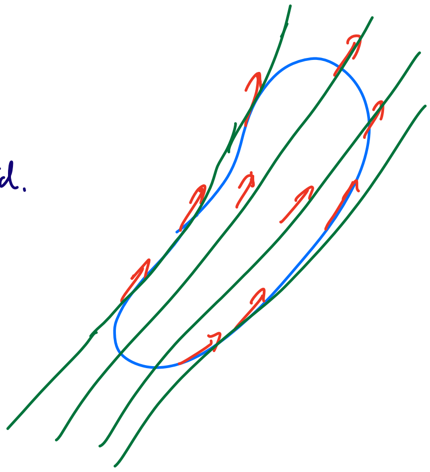
True for any control volume, and hence

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{u}) = 0} \qquad (B)$$

Material Parcel

Material parcel follows fluid elements

- can think of as "tagged" particles
- Velocity of boundary same as that of fluid.



For material parcel, mass is conserved

$$\frac{d}{dt}\oint_{V(t)} \rho\, dV = \frac{dM}{dt} = 0$$

Lagrangian co-ordinates

Associated with the material parcel, we can define "material" or Lagrangian coordinates

$$\underset{\sim}{x}_0 = (x_0, y_0, z_0)$$

where $\underset{\sim}{x}_0$ is the position of a given parcel at some time $t = t_0$.

Can think of $\underset{\sim}{x}_0$ as a "tag" that follows parcels around.

This can be compared to Eulerian coordinates

$$\underset{\sim}{x} = (x, y, z)$$

which refer to the position in some inertial reference frame.

$\underset{\sim}{x}_0$ is constant for a given fluid element

$\underset{\sim}{x}$ is constant for a given position in space.

What does $\left.\frac{\partial \underset{\sim}{x}}{\partial t}\right|_{\underset{\sim}{x}_0}$ mean?

Lagrangian Derivative

Need to convert between the Eulerian & Lagrangian viewpoints

By definition the velocity is defined,

$$\underset{\sim}{u} = (u, v, w) = \left.\frac{\partial \underset{\sim}{x}}{\partial t}\right|_{\underset{\sim}{x}_0}$$

i.e., the velocity is the rate of change of position following a parcel.

We can use this to express the material derivative in Eulerian coordinates

Consider a variable χ.

May write this as a function of $\underset{\sim}{x}$ $\quad \chi = \chi(\underset{\sim}{x}, t)$

or as a function of $\underset{\sim}{x}_0$ $\quad \chi = \chi(\underset{\sim}{x}_0, t)$

By the chain rule, we have

$$\left.\frac{\partial \chi}{\partial t}\right|_{\underset{\sim}{x}_0} = \left.\frac{\partial \chi}{\partial t}\right|_{\underset{\sim}{x}} \left.\frac{\partial t}{\partial t}\right|_{\underset{\sim}{x}_0} + \left.\frac{\partial \chi}{\partial x}\right|_{\underset{\sim}{x}} \left.\frac{\partial x}{\partial t}\right|_{\underset{\sim}{x}_0} + \left.\frac{\partial \chi}{\partial y}\right|_{\underset{\sim}{x}} \left.\frac{\partial y}{\partial t}\right|_{\underset{\sim}{x}_0} + \left.\frac{\partial \chi}{\partial z}\right|_{\underset{\sim}{x}} \left.\frac{\partial z}{\partial t}\right|_{\underset{\sim}{x}_0}$$

$$= \left.\frac{\partial \chi}{\partial t}\right|_{\underset{\sim}{x}} + u \left.\frac{\partial \chi}{\partial x}\right|_{\underset{\sim}{x}} + v \left.\frac{\partial \chi}{\partial y}\right|_{\underset{\sim}{x}} + w \left.\frac{\partial \chi}{\partial z}\right|_{\underset{\sim}{x}}$$

$$= \frac{\partial \chi}{\partial t} + \underset{\sim}{u} \cdot \nabla \chi$$

where all partial derivatives are taken at constant $\underset{\sim}{x}$.

Since this transformation is so useful, we give it a special symbol:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla$$

The Lagrangian derivative.

Conservation of momentum

Newton's law applied to a fluid parcel:

$$\underbrace{\frac{d}{dt}\oint_{V(t)} \rho \underset{\sim}{u}\, dV}_{\text{parcel momentum}} = \underbrace{\sum \underset{\sim}{F}}_{\text{sum of forces}}$$

Consider the LHS. Can't move derivative through the integral because the limits depend on time.

Transform to Lagrangian coordinates $(x,y,z) \longmapsto (x_0, y_0, z_0)$

$$\oint_{V(t)} \rho \underset{\sim}{u}\, dV = \oint_{V_0} \rho \underset{\sim}{u}\, J(\underset{\sim}{x}, \underset{\sim}{x_0})\, dV$$

$J(\underset{\sim}{x_0}, \underset{\sim}{x})$ is the Jacobian for this transformation

what is this?

Now integration limits are independent of time:

$$\text{LHS} = \frac{d}{dt}\oint_{V_0} \rho \underset{\sim}{u}\, J(\underset{\sim}{x}, \underset{\sim}{x_0})\, dV = \oint_{V_0} \frac{\partial}{\partial t}\left\{ \rho \underset{\sim}{u}\, J(\underset{\sim}{x}, \underset{\sim}{x_0}) \right\}\Big|_{\underset{\sim}{x_0}} dV$$

$$= \oint_{V_0} \frac{\partial \underset{\sim}{u}}{\partial t}\Big|_{\underset{\sim}{x_0}} \rho\, J(\underset{\sim}{x}, \underset{\sim}{x_0})\, dV + \oint \underset{\sim}{u}\, \frac{\partial}{\partial t}(\rho J)\Big|_{\underset{\sim}{x_0}} dV$$

First term: Jacobian unaffected → simply transform back!

$$\oint_{V_0} \frac{\partial \underset{\sim}{u}}{\partial t}\Big|_{\underset{\sim}{x_0}} \rho\, J(\underset{\sim}{x}, \underset{\sim}{x_0})\, dV = \oint_{V(t)} \frac{D\underset{\sim}{u}}{Dt}\, \rho\, dV$$

Second term: Consider mass of material element

By definition $\frac{d}{dt}\oint_{V(t)} \rho\, dV = 0$

In Lagrangian coordinate:

$$\frac{d}{dt}\oint_{V(t)} \rho \, dV = \frac{d}{dt}\oint_{V_0} \rho J \, dV = \oint_{V_0} \left.\frac{\partial(\rho J)}{\partial t}\right|_{\underset{\sim}{x}_0} dV = 0$$

Since V_0 is arbitrary, implies $\left.\frac{\partial(\rho J)}{\partial t}\right|_{\underset{\sim}{x}_0} = 0$

Second term above is zero!

Thus, we have,

$$\frac{d}{dt}\oint_{V(t)} \rho \underset{\sim}{u} \, dV = \oint \frac{D\underset{\sim}{u}}{Dt} \rho \, dV$$

Now consider the right hand side: sum of forces.

Now tally the forces on the air parcel

Divide into $\underbrace{\text{body forces}}_{\uparrow \text{ act throughout fluid}}$ & $\underbrace{\text{surface forces}}_{\uparrow \text{ act at interfaces between fluid elements}}$

What are the forces acting on a fluid?

Relevant forces for atmosphere:

- gravitational force (body force

$\left.\begin{array}{l}\text{- pressure} \\ \text{- stress (friction)}\end{array}\right\}$ (surface forces)

Gravity:

$$\underset{\sim}{F_g} = \oint_{V_t} \rho \underset{\sim}{g} \, dV$$

gravitational force per unit mass

$$\underset{\sim}{F_g} = -\oint_{V_t} \rho \nabla \Phi_g \, dV$$

gravitational potential

pressure force: acts at surface bounding parcel. Points inwards normal to surface

What is pressure?

$$\underset{\sim}{F}_p = - \oint_{\partial V_t} p \, \underset{\sim}{n} \, dS$$

By divergence theorem

$$\underset{\sim}{F}_p = - \int_{V_t} \nabla p \, dV$$

Also have surface forces due to friction. Can represent by viscous stress tensor P_{ij}. i^{th} component of stress on surface oriented with normal n_j

$$F_{\nu,i} = \sum_j P_{ij} n_j$$

Viscous force on parcel:

$$\underset{\sim}{F}_\nu = \oint_{\partial V_t} \underset{\sim}{n} \cdot \underset{\approx}{P} \, dS$$

$$= \oint_{V_t} \nabla \cdot \underset{\approx}{P} \, dV \qquad \text{(divergence theorem)}$$

Putting all this together:

Viscous stress tensor

$$\oint \rho \frac{D\underset{\sim}{u}}{Dt} dV = \oint_{V_t} -\rho \nabla \Phi_g - \nabla p + \nabla \cdot \underset{\approx}{P} \, dV$$

Since V_t is arbitrary, the integrands must be equal:

For now summarise viscous stresses as $\underset{\sim}{F}_{fric}$

$$\frac{D\underset{\sim}{u}}{Dt} = -\nabla \Phi_g - \frac{1}{\rho} \nabla p + \underset{\sim}{F}_{fric}$$

Rotation

Earth's rotation gives rise to additional forces

Centrifugal: $\underset{\sim}{F}_{cent} = -\underset{\sim}{\Omega} \times \underset{\sim}{\Omega} \times \underset{\sim}{r}$

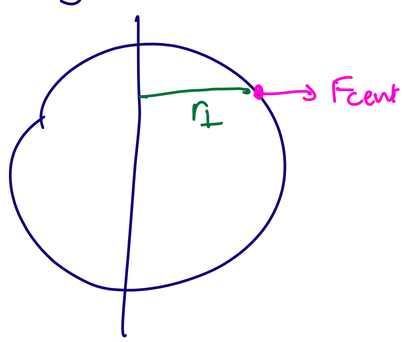
rotation vector

→ depends on position only

Coriolis $\underset{\sim}{F}_{cor} = -2\underset{\sim}{\Omega} \times \underset{\sim}{u}$

→ depends on velocity, acts at right angles.

$\underset{\sim}{F}_{cent}$ points radially outwards:



Magnitude of centrifugal force $= \dfrac{v^2}{r_\perp} = \dfrac{\Omega^2 r_\perp^2}{r_\perp} = \Omega^2 r_\perp$

How do we deal with these forces?

Consider stationary object on Earth.

Forces: gravity, centrifugal, normal force

How can we make these forces balance?

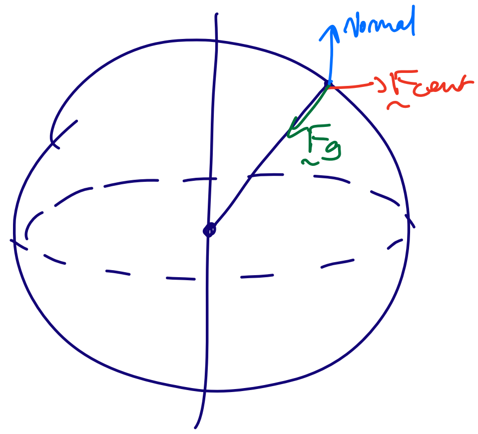
Ask about forces

Bulging of Earth

- The force associated with $-\nabla\Phi$ does not point to the centre for the Earth

- But it does correspond to local "down" as measured by a plumb bob.

- Earth's shape has ajusted to bulge at the equator so that the normal force from the surface is opposite to the combination gravity & centrifugal force. (over the ocean at least).



Makes sense to combine centrifugal force and gravitational force.

Can express centrifugal force as gradient of potential:

$$\underline{F}_{cent} = \nabla\left(\frac{\Omega^2 r_\perp^2}{2}\right)$$

Can easily verify that:

$$|\underline{F}_{cent}| = |\Omega^2 r_\perp| \quad , \quad \hat{\underset{\sim}{F}}_{cent} = \hat{\underset{\sim}{r}}_\perp$$

We can therefore define a modified potential such that

$$\underset{\sim}{F}_g + \underline{F}_{cent} = -\nabla\underline{\Phi} = -\nabla\left[\Phi_g - \frac{\Omega^2 r_\perp^2}{2}\right]$$

The momentum equation may then be written

$$\boxed{\frac{D\underline{u}}{Dt} = -\nabla\underline{\Phi} - 2\underline{\Omega}\times\underline{u} - \frac{1}{\rho}\nabla p + \underset{\sim}{F_f}} \qquad (C)$$

In this equation $\underset{\sim}{u}$ is measured relative to the rotating Earth.

Spherical Co-ordinates

- Convenient to write equations in spherical co-ordinates

$$(x, y, z) \longmapsto (\lambda, \phi, r)$$

radius (dist. from centre of Earth)

latitude   longitude

- But Earth not exactly spherical   What do we do?
- Make the approximation that surfaces of constant Φ are true spheres.

$$\Rightarrow \quad -\nabla\underline{\Phi} = -g\,\hat{\underset{\sim}{k}}$$

gravitational + centrifugal force per unit mass

unit vector in the radial direction ("up")

Can be easily seen that,

$$u = r\cos\phi \frac{d\lambda}{dt} \quad , \quad v = r\frac{d\phi}{dt} \quad , \quad w = \frac{dr}{dt}$$

Also:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{d\lambda}{dt}\frac{\partial}{\partial\lambda} + \frac{d\phi}{dt}\frac{d}{d\phi} + \frac{dr}{dt}\frac{d}{dr}$$

$$= \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\delta\phi} + w\frac{\partial}{\partial r}$$

For flow on Earth $r = R_e + z$

radius of Earth

Furthermore, since $R_e \gg z$ for the atmosphere, we can make a series of approximations that simplify the equations.

This results in a set of equations called the "primitive equations"

see Vallis Ch 2.2.

$\Rightarrow$ see slides for actual formulas.

Divergence

Under the shallow fluid approx

$$\nabla \cdot \underline{u} = \frac{1}{R_e\cos\phi}\left\{ \frac{\partial u}{\partial\lambda} + \frac{\partial v\cos\phi}{\partial\phi} \right\} + \frac{\partial w}{\partial z}$$

$r \to R_e$ by shallow fluid approx.

Thermodynamic equation

-> First law of thermodynamics represents conservation of energy at micro-scales

- Q: How does this interact with the momentum eq$^{\underline{n}}$ and conservation of macro-scale kinetic energy?

  See: Salmon, R (1998), Lectures on GFD for nice treatment.

We just take as given the thermodynamic equation

$$\underset{\text{heating rate}}{Q} = \underset{\substack{\text{change}\\ \text{in internal}\\ \text{energy}}}{\frac{De}{Dt}} + \underset{\text{work}}{p\frac{D\alpha}{Dt}}$$

For fluids with constant composition, can express internal energy in terms of any two co-ordinates. e.g.,

$$e = e(T, \alpha) \qquad [\text{for ideal gas } e = e(T) \text{ only}]$$

By chain rule:

$$\frac{De}{Dt} = \left(\frac{\partial e}{\partial T}\right)_\alpha \frac{DT}{Dt} + \left(\frac{\partial e}{\partial \alpha}\right)_T \frac{D\alpha}{Dt}$$

$$Q = \left(\frac{\partial e}{\partial T}\right)_\alpha \frac{DT}{Dt} + \left[\left(\frac{\partial e}{\partial \alpha}\right)_T + p\right] \frac{D\alpha}{Dt}$$

at constant volume, we have

$$Q = \left(\frac{\partial e}{\partial T}\right)_\alpha \frac{DT}{Dt}$$

Define $c_v = \left(\frac{\partial e}{\partial T}\right)_\alpha$

$$\Rightarrow \quad Q = c_v \frac{dT}{dt} + p \frac{D\alpha}{Dt}$$

specific heat capacity at const. volume

Over atmospheric temperature ranges, c_v is roughly constant.

Note that the ideal gas law

$$p\alpha = R_d T$$

allows us to write

$$\alpha \frac{Dp}{Dt} + p \frac{D\alpha}{Dt} = R_d \frac{DT}{Dt}$$

Substitute into thermodynamic equation

$$\boxed{Q = c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt}} \qquad (D)$$

$$= c_v \frac{DT}{Dt} + R_d \frac{DT}{Dt} - \alpha \frac{Dp}{Dt}$$

$$Q = (c_v + R_d) \frac{DT}{Dt} - \alpha \frac{Dp}{Dt}$$

$c_p = c_v + R_d$
is the specific heat
capacity at constant pressure.

This equation is particularly useful in pressure coordinates since $\frac{Dp}{Dt} = \omega$ is the vertical velocity.

Equations (A-D) represent a complete set to describe a dry atmosphere with the specification of appropriate boundary conditions and forcing terms (e.g., q)

- In addition generally require equation for humidity

Useful approximations & transformations

Tangent plane approx.

take $x \sim R\cos\phi_0 \lambda$
$y \sim R_e(\phi - \phi_0)$

- Recovers familiar f-plane or β-plane equations

Pressure co-ordinates

Transform equations into new co-ordinates

$$(x, y, z, t) \longmapsto (x', y', p, t')$$

Need to transform terms in eq^{n}s

Lagrangian derivative:

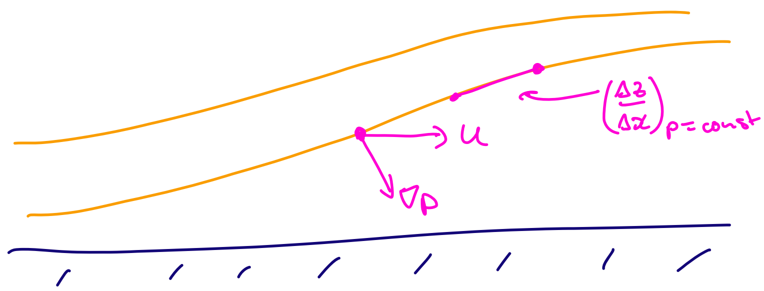
$$\frac{D}{Dt} = \frac{\partial}{\partial t'} + u\frac{\partial}{\partial x'} + v\frac{\partial}{\partial y'} + \omega\frac{\partial}{\partial p}$$

$$\omega = \frac{Dp}{Dt}$$

Note that $u = \frac{dx'}{dt} = \frac{dx}{dt}$, $v = \frac{dy'}{dt} = \frac{dy}{dt}$

u and v remain as horizontal velocities!

But we take derivatives along constant pressure surfaces



Need to transform derivatives in equations

The new coordinates are related to the old by:

$$x = x'$$
$$y = y'$$
$$\frac{\partial p}{\partial z} = -\rho g \qquad \text{(hydrostatic balance)}$$

Consider an arbitrary variable γ,

$$\frac{\partial \gamma}{\partial x'} = \frac{\partial \gamma}{\partial x}\underbrace{\frac{\partial x}{\partial x'}}_{=1} + \cancelto{0}{\frac{\partial \gamma}{\partial y}\frac{\partial y}{\partial x'}} + \frac{\partial \gamma}{\partial z}\frac{\partial z}{\partial x'} + \cancelto{0}{\frac{\partial \gamma}{\partial t}\frac{\partial t}{\partial x'}}$$

derivative at const. pressure

$$\frac{\partial \gamma}{\partial x'} = \frac{\partial \gamma}{\partial x} + \frac{\partial \gamma}{\partial z}\frac{\partial z}{\partial x'}$$

slope of isobar

Take $\gamma = p$ (pressure)

$$\frac{\partial p}{\partial x'} = 0 \qquad \text{(derivative at const. pressure.)}$$

$$\Rightarrow \quad 0 = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial z}\frac{\partial z}{\partial x'}$$

$$\frac{\partial p}{\partial x} = -\rho g \frac{\partial z}{\partial x'}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial \Phi}{\partial x'} \qquad (\Phi = gz)$$

Thus, our tangent-plane equations may be written,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p} = fv - \frac{\partial \Phi}{\partial x} + F_x$$

where we have dropped the primes, and horizontal derivatives are taken at constant pressure.

Note that we also have

$$\frac{\partial \Phi}{\partial p} = -\alpha \qquad \text{(Hydrostatic balance)}$$

Continuity in pressure coordinates

- Can transform the equation similarly
- But simpler to rederive applying hydrostatic balance

Remember $$M = \int_{V(t)} \rho \, dx\,dy\,dz$$

Hydrostatic balance: $$\rho dz = \frac{1}{g} dp$$

$$M = \int_{V(t)} \frac{1}{g} \, dx\,dy\,dp$$

"Density" in pressure coordinates is $\frac{1}{g}$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{1}{g}\right) + \nabla_p \cdot \left(\frac{\underline{u}}{g}\right) = 0$$

$$\Rightarrow \quad \nabla_h \cdot \underset{\sim}{u} + \frac{\partial \omega}{\partial p} = 0$$

Under the tangent plane approx:

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Can show this more rigorously!

Definition of the streamfunction

Useful to define streamfunction ψ such that the difference in ψ between two levels is equal to the mass flux between the levels

That is,

$$\psi(z_2) - \psi(z_1) = R_e \cos\phi \int_0^{2\pi} \int_{z_1}^{z_2} \rho v \, dz \, d\lambda$$

Since $dz = -\rho g dp$, can write in pressure coordinates

$$\psi(p_1) - \psi(p_2) = R_e \cos\phi \int_0^{2\pi} \int_{p_1}^{p_2} v \frac{dp}{g} \, d\lambda$$

Differentiating in pressure

$$g \frac{\partial \psi}{\partial p} = R_e \cos\phi \int_0^{2\pi} v \, d\lambda$$

$$\frac{g}{2\pi R_e \cos\phi} = [v]$$

where $[v]$ is a zonal mean.

Remember, continuity equation may be written

$$\frac{1}{R_e \cos\phi} \left\{ \frac{\partial \rho u}{\partial \lambda} + \frac{\partial \rho v \cos\phi}{\partial \phi} \right\} + \frac{\partial \omega}{\partial p} = 0$$

Integrate in longitude and multiply by $R_e \cos\phi$

$$\frac{\partial}{\partial \phi} \left(\int_0^{2\pi} \rho v \, d\lambda \, R_e \cos\phi \right) + R_e \cos\phi \frac{\partial}{\partial p} \left\{ \int_0^{2\pi} \omega \, d\lambda \right\} = 0$$

Comparing to above:

$$\frac{\partial}{\partial \phi} \left\{ g \frac{\partial \psi}{\partial p} \right\} + R_e \cos\phi \frac{\partial}{\partial p} \left\{ \int_0^{2\pi} \omega \, d\lambda \right\} = 0$$

$$\frac{\partial}{\partial p} \left\{ g \frac{\partial \psi}{\partial \phi} + R_e \cos\phi \int_0^{2\pi} \omega \, d\lambda \right\} = 0$$

Assume $\psi = \omega = 0$ at surface:

$$\frac{g}{2\pi R_e \cos\phi} \frac{\partial \psi}{\partial \phi} = \frac{-1}{2\pi} \int_0^{2\pi} \omega \, d\lambda$$

$$\frac{g}{2\pi R_e \cos\phi} \frac{\partial \psi}{\partial \phi} = [\omega]$$

$\Rightarrow$ The difference in ψ across latitudes corresponds to the vertical mass flux between those latitudes.

Next Lecture: Use these equations to think about the basic causes of the general circulation